1) Let $S \subseteq \mathbb{R}$ be nonempty and bounded above by $M$. Then by the completeness axiom, $S$ has a least upper bound, say $L = \text{sup}(S)$. Since $M$ is an upper bound of $S$, $L \leq M$.

Suppose that $L < M$. Let $\varepsilon = \frac{M - L}{2}$. Then there exists $x \in S$ such that $x > M - \varepsilon = M - \frac{M - L}{2} = \frac{M + L}{2} > \frac{L + L}{2} = L$, contradicting the fact that $L$ is an upper bound for $S$. Therefore $L = M$.

2) Prove $n! \leq 2\left(\frac{n}{2}\right)^n \forall n \in \mathbb{N}$

Proof by induction.

For $n = 1$, $1! \leq 2\left(\frac{1}{2}\right)^1 \iff 1 \leq 1$ is true.

Suppose statement is true for a given $n$. Consider $n+1$.

$$(n+1)! = (n+1) n! \leq (n+1) 2\left(\frac{n}{2}\right)^n$$

by induction assumption.

Thus it suffices to prove $(n+1) 2\left(\frac{n}{2}\right)^n \leq 2\left(\frac{n+1}{2}\right)^{n+1}$

$$\Leftrightarrow \frac{(n+1)n^n}{2^n} \leq \frac{(n+1)(n+1)^n}{2 \cdot 2^n} \iff 2 \leq \left(\frac{n+1}{n}\right)^n \iff 2 \leq (1 + \frac{1}{n})^n \star$$

Now, since $\left(1 + \frac{1}{n}\right)^n$ is increasing, every term is greater than the first term $\left(1 + \frac{1}{n}\right)^n$. Thus $\star$ is true. QED.
3.) \[ \lim \limits_{n \to \infty} \frac{2n+1}{n+3} = \lim \limits_{n \to \infty} \frac{2 + \frac{1}{n}}{1 + \frac{3}{n}} = \lim \limits_{n \to \infty} \frac{2 + \frac{1}{n}}{1} = 2 \]

Let \( \epsilon > 0 \) be given.

We want \[ \left| \frac{2n+1}{n+3} - 2 \right| < \epsilon \iff \left| \frac{2n+1 - 2(n+3)}{n+3} \right| < \epsilon \]

\[ \iff \left| \frac{-5}{n+3} \right| < \epsilon \iff \frac{5}{n+3} < \epsilon \iff n+3 > \frac{5}{\epsilon} \iff n > \frac{5}{\epsilon} - 3 \]

Set \( N = \left\lceil \frac{5}{\epsilon} - 3 \right\rceil + 1 \). Then, if \( n > N \) we have \( n > \frac{5}{\epsilon} - 3 \)

and so by \( \left| \frac{2n+1}{n+3} - 2 \right| < \epsilon \) \( \square \) QED

4.) \[ \lim \limits_{x \to 2} \frac{3x^2 - 12}{x - 2} = \lim \limits_{x \to 2} \frac{3(x^2 - 4)}{(x-2)} = \lim \limits_{x \to 2} \frac{3(x+2)}{x-2} = 12 \]

We want \[ \left| \frac{3x^2 - 12}{x - 2} - 12 \right| < \epsilon \iff \left| 3(x+2) - 12 \right| < \epsilon \]

\[ \iff \left| 3x - 6 \right| < \epsilon \iff \left| x - 2 \right| < \frac{\epsilon}{3} \]

Set \( \delta = \frac{\epsilon}{3} \). Then, if \( 0 < \left| x - 2 \right| < \delta \) it follows from \( \star \) that \( \left| \frac{3x^2 - 12}{x - 2} - 12 \right| < \epsilon \)

\( QED \).
5] If \( \lim_{x \to a^+} f(x) = L > 0 \) then for any \( \varepsilon > 0 \), \( \exists \delta > 0 \) such that if \( a < x < a + \delta \), then \( |f(x) - L| < \varepsilon \). In particular, letting \( \varepsilon = \frac{L}{2} \), \( \exists \delta > 0 \) if \( a < x < a + \delta \), then \( |f(x) - L| < \frac{L}{2} \). But this implies
\[
f(x) - L > -\frac{L}{2} \implies f(x) > \frac{L}{2} > 0
\]
for any \( x \in (a, a + \delta) \). \( \Box \).

6] Suppose \( a_n \to a \), \( a \neq a \) \( \forall n \in \mathbb{N} \) and \( \lim_{x \to a} f(x) = L \).
Let \( \varepsilon > 0 \) be given. Then \( \exists \delta > 0 \) such that if \( 0 < |x - a| < \delta \) then \( |f(x) - L| < \varepsilon \). Now, since \( a_n \to a \), \( \exists N \in \mathbb{N} \) such that if \( n > N \) then \( |a_n - a| < \delta \). Since \( a \neq a \) \( \forall n \in \mathbb{N} \) we have in fact \( 0 < |a_n - a| < \delta \) for \( n > N \).
Thus (letting \( x = a_n \) above) we obtain
\[
|f(a_n) - L| < \varepsilon \quad \text{for} \quad n > N
\]
\[\therefore \lim_{n \to \infty} f(a_n) = L\].
7) Suppose that $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ and that

\[(k) \quad f(x) \leq g(x) \leq h(x) \quad \text{for} \quad 0 < |x - a| < \delta_2, \text{ say.}\]

Then $\lim_{x \to a} g(x) = L$.

**Proof.** Let $\varepsilon > 0$ be given. Since $\lim_{x \to a} f(x) = L$, there exists $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \varepsilon$, in particular, $f(x) > L - \varepsilon$. (1)

Since $\lim_{x \to a} h(x) = L$, there exists $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$, then $|h(x) - L| < \varepsilon$, in particular, $h(x) < L + \varepsilon$. (2)

Set $\delta = \min(\delta_1, \delta_2, \delta_3)$. Then if $0 < |x - a| < \delta_3$, we have

$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon$, and so

by (1) \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow

$|g(x) - L| < \varepsilon$.

QED.

8) Proof by contradiction. Suppose that $\sqrt{2} \in \mathbb{Q}$, say $\sqrt{2} = \frac{a}{b}$, where $a, b$ have no common factor other than $\pm 1$.

$\Rightarrow b \sqrt{2} = a \quad \Rightarrow \quad b^2 \cdot 2 = a^2 \quad \Rightarrow \quad 2$ is a factor of $a$, say $a = 2a$.

Then $2b^2 = 8a^3 \Rightarrow b^2 = 4a^3 \Rightarrow$ 2 is a factor of $b$, a contradiction (since $a, b$ have no common prime factor).

$\therefore \sqrt{2}$ is irrational.