Solutions to HW 7:

1) Let \( \varepsilon > 0 \) be given. Since \( a_n \to L \), \( \forall N \in \mathbb{N} \), \( n > N \Rightarrow |a_n - L| < \varepsilon \).

Now, since \( \{a_n\} \) is an increasing sequence, \( a_k \leq a_N \forall k \leq N \). Set \( k = N \). Then for \( k > K \), \( a_k \geq K = N \) and so \( |a_k - L| < \varepsilon \). \( \therefore \varepsilon > 0 \).

2) If \( \{a_n\} \) converges to \( L \), then so does every subsequence of \( \{a_n\} \), by problem 1. Therefore \( L \) is the only cluster point.

3) If \( a_n \to L \) as \( n \to \infty \), then \( \varepsilon > 0 \) such that \( \forall N \in \mathbb{N} \), \( \forall n > N \) with \( |a_n - L| < \varepsilon \). Let \( \{a_{n_k}\} \) be the subsequence consisting of all terms of the original sequence satisfying \( * \). Since \( \{a_n\} \) is bounded, so is \( \{a_{n_k}\} \) and so by Bolzano-Weierstrass, \( \{a_{n_k}\} \) has a cluster pt \( C \). Clearly \( C \neq L \) by \( * \). Then \( \{a_{n_k}\} \) has a subsequence converging to \( C \), but this subsequence is also a subsequence of \( \{a_n\} \) and so \( C \) is a cluster pt of \( \{a_n\} \), a contradiction.

4) Since \( \{T_n^2\} \) converges it is Cauchy so given \( \varepsilon > 0 \), \( \exists N \in \mathbb{N} \) s.t. \( |T_n - T_m| < \varepsilon \).

Now \( |S_n - S_m| = \sum_{k=m}^{n} |a_k| \leq \sum_{k=m}^{n} |a_k| = |T_n - T_m| \).

Thus if \( n \geq m \) then \( |S_n - S_m| \leq |T_n - T_m| < \varepsilon \), so \( \{S_n\} \) is Cauchy.

Therefore, \( \{S_n\} \) converges. If a series converges absolutely, then it converges.

5) a) Since \( f \) is unit (and on \( (a, b) \)), given \( \varepsilon > 0 \), \( \exists \delta > 0 \) s.t. \( |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \). Since \( \{a_n\} \) converges, it is Cauchy, so given \( \exists N \in \mathbb{N} \), \( n, m > N \Rightarrow |a_n - a_m| < \delta \). We assume \( a \in (a, b) \) \( \forall n \in \mathbb{N} \).

Then for \( n, m > N \) we have \( |x - y| < \delta \Rightarrow |f(a_n) - f(a_m)| < \varepsilon \).

\( \therefore \{f(a_n)\} \) is Cauchy, and so it converges.
Let \( \{a_n\} \) be a sequence with \( a_n \to b \) as \( n \to \infty \). By (a), \( f(a_n) \) converges. Say \( f(a_n) \to L \) as \( n \to \infty \). Then given \( \varepsilon > 0 \), \( \exists N_1 \) such that \( n > N_1 \Rightarrow |f(a_n) - L| < \varepsilon/2 \). Since \( f \) is uniformly continuous, \( \exists N_2 \) such that \( n > N_2 \Rightarrow |f(x) - f(y)| < \varepsilon/2 \). Finally, since \( a_n \to b \), \( \exists N_2 \) such that \( n > N_2 \Rightarrow |a_n - b| < \delta \). Suppose \( n > \max(N_1, N_2) \), then since and that \( b - \delta < x < b \). Then since \( a_n, x \in (b - \delta, b) \) we have \( |a_n - x| < \delta \) and so by \( \delta \), \( |f(a_n) - f(x)| < \varepsilon/2 \). Thus

\[
|f(x) - L| = |f(x) - f(a_n) + f(a_n) - L| \leq |f(x) - f(a_n)| + |f(a_n) - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\]

for \( b - \delta < x < b \). QED.

5) Let \( x \in (a, b) \). Set \( \delta = \min(\frac{b-x}{a}, \frac{x-a}{b}) \). Then \( (x-\delta, x+\delta) \) is a NBD of \( x \) with \( (x-\delta, x+\delta) \subset (a, b) \). i.e. \( (a, b) \) is open.

b) Let \( \mathcal{U} = \{ U_x \}_{x \in [a, b]} \) be a covering of \([a, b]\) by open sets. For any \( x \in [a, b] \), \( \exists U_x \in \mathcal{U} \) with \( x \in U_x \), since \( \mathcal{U} \) covers \([a, b]\). Since \( U_x \) is open, \( \exists \) open interval \( I_x \) with \( x \in I_x \subset U_x \). Then \( \mathcal{I} = \{ I_x \}_{x \in [a, b]} \) is a covering of \([a, b]\) by open intervals. By Heine-Borel, \( \mathcal{I} \) has a finite subcovering \( \{ I_{x_1}, I_{x_2}, \ldots, I_{x_k} \} \). Then \( \{ U_{x_1}, U_{x_2}, \ldots, U_{x_k} \} \) is a finite subcovering of \( \mathcal{U} \), since

\[
[a, b] = \bigcup_{j=1}^{k} I_{x_j} \subseteq \bigcup_{i=1}^{k} U_{x_k}.
\]