Let \( f : S \to T \) (where \( S \) and \( T \) are subsets of \( \mathbb{R} \)) be a function with domain \( S \) and range \( T \). Prove that \( f \) has an inverse \( g : T \to S \) if and only if \( f \) is one-to-one on \( S \). (Verify all 3 properties of \( g \) under the inverse function definition.)

Suppose \( f : S \to T \) and \( f \) has an inverse \( g : T \to S \).

Suppose \( x_1, x_2 \in S \) with \( f(x_1) = f(x_2) \). Then \( g \circ f(x_1) = g \circ f(x_2) \), \( x_1 = x_2 \) \( \Rightarrow \) \( f \) is one-to-one on \( S \).

Now prove the converse:

Suppose \( f : S \to T \) is one-to-one function. Define \( g : T \to S \) by setting \( g(y) = x \) where \( x \) is the unique \( x \) in \( S \) such that \( f(x) = y \). Then \( g \circ f(x) = g(y) = x \) \( \forall x \in S \) and \( f \circ g(y) = f(x) = y \) \( \forall y \in T \) \( \Rightarrow \) \( g \) is an inverse of \( f \).

2. Let \( f \) be a real valued function defined on \( I = (a, b) \). Suppose that for any monotone increasing sequence \( \{a_n\} \) with \( a \leq a_n \leq b \) \( \forall n \to \infty \), we have \( f(a_n) \to f(b) \). Prove, for \( \forall \varepsilon > 0 \), that \( f \) is left continuous at \( b \). (Proof by contradiction as for The 2.4d)

Proof by contradiction: Suppose that \( \lim_{x \to b^-} f(x) \neq f(b) \)

Then \( \exists \varepsilon > 0 \) \( \exists \delta > 0 \) \( \exists x \in (b - \delta, b) \) \( \exists \delta | f(x) - f(b) \| = \varepsilon \).

Now define \( \{a_n\} \) inductively as follows: for \( \delta = \min \left( \frac{1}{n}, b - a_{n-1} \right) \), \( \forall n \exists a_n \in (b - \delta, a_{n-1}) \) and \( |f(a_n) - f(b)| = \varepsilon \).

However, \( b - \frac{1}{n} < a_n < b \), then \( a_n \to b \) \( a \to b \) by the squeeze law.

Also, \( b - a_n < b - a_{n-1} \Rightarrow a_n > a_{n-1} \Rightarrow \{a_n\} \) is a monotone increasing sequence and converges to \( b \). It follows that \( f(a_n) \to f(b) \), a contradiction.

\( \therefore f(x) = f(b) \Rightarrow f \) is left continuous at \( b \).
3. Let I be an interval of real numbers. A subset S of I is called a dense subset of I if every point of I is a limit point of S.

(a) Prove that S is a dense subset of I if every open subinterval J of I contains at least one point of S. (In particular, the set of rationals in I and the set of irrationals in I are both dense subsets of I.)

=>
Suppose S is a dense subset of I, and let J be an open subinterval of I. Since every point of I is a limit point of S, every pt. of J is a limit pt. of S. Let a be the mid-point of J and let \( \delta = \frac{|J|}{2} \). Then the \( \delta \)-Nbd of \( a \), i.e., the interval \( J \), contains a point of S.

<=
Suppose every open subinterval of I contains at least one point of S. Prove that S is a dense subset of I. i.e., prove that every point of I is a limit point of S.

Let a be any point in I. Then \( \forall \delta > 0 \), the interval \( (a-\delta, a+\delta) \) will include some subinterval J of I with a in J. \( \Rightarrow \) (a-\( \delta \), a+\( \delta \)) contains at least one point of S other than \( a \) \( \Rightarrow \) Every del. Nbd of \( a \) contains a point of S \( \Rightarrow \) a is a limit point of S.

Since any point of I is a limit point of S, S is a dense subset of I.
3(b) If \( S \) is a dense subset of \( I \) and \( a \in I \), prove that there is a sequence \( \{x_n\} \) of points in \( S \) such that \( x_n \to a \) as \( n \to \infty \).

* Suppose \( S \) is a dense subset of \( I \), then every point of \( I \) is a limit point of \( S \).
  
  Let \( a \in I \), then \( a \) is a limit point of \( S \).
  
  \[
  \Rightarrow \text{every NEE of } a \text{ contains a point of } S.
  \]

  Let \( x_n \) be a point in \( S \text{ for } n \in \mathbb{N} \) with \( x_n \in S \).
  
  Then \( x_n \to a \) as \( n \to \infty \).

3(c) Suppose \( f, g \) are continuous functions on \( I \) that agree on a dense subset \( S \) of \( I \), that is, \( f(x) = g(x) \) for \( x \in S \).

Prove that \( f(x) = g(x) \) for all \( x \in I \).

* Let \( a \) be any point in \( I \). Since \( S \) is a dense subset of \( I \), \( a \) is a limit point of \( S \).

  \[
  \exists \text{ a sequence } \{x_n\} \text{ of points in } S \text{ with } x_n \to a \text{ as } n \to \infty .
  \]

  Since \( f, g \) are continuous on \( I \), \( f, g \) are continuous at \( a \).

  \[
  \Rightarrow f(a) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = g(a).
  \]

  Since \( x_n \in S \),

  \[
  \Rightarrow f(x) = g(x) \text{ for all } x \in I.
  \]