1] Prove any open interval \((a, b)\) of real numbers contains a rational number.

**Proof:** By the Archimedean property there exists \(n \in \mathbb{N} \) with \(n > \frac{1}{b-a}\), that is, \(\frac{1}{n} < b-a\). Consider the set

\[ S = \{ k \in \mathbb{Z} : k/n < b \} \]

Set \(S\) is bounded above by \(nb\) and so by the Max Ell Principle it has a max \(m\). We claim that \(m/n \in (a, b)\). Certainly \(m/n < b\), since \(m \in S\). If \(m/n \leq a\) then \(m+1/n < a+(b-a) = b\) and so \(m+1\in S\) contradicting the maximality of \(m\). Therefore, \(m/n > a\).

2] See 1.7B7, B8 on following pages.

3] Prove \(e\) is irrational.

**Proof by contradiction.** Suppose \(e\) is rational, say \(e = \frac{p}{q}\) with \(p, q \in \mathbb{N}\). Then \(\frac{1}{e} = \sum_{k=0}^{\infty} \frac{1}{k!}\)

\[ e! \cdot \frac{1}{e} = \sum_{k=0}^{\infty} \frac{1}{k!} = \sum_{k=0}^{\infty} \frac{q!}{k!} + \sum_{k=q+1}^{\infty} \frac{q!}{k!} \]

\[ \Rightarrow (q-1)! \cdot p = e! + \frac{q!}{2!} + ... + \frac{q!}{q!} + \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + ... \]

Now for \(k=0, \ldots, q\), \(\frac{q!}{k!} \in \mathbb{N}\) and thus

\[ \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + ... = (q-1)! \cdot p - q! - ... - \frac{q!}{e!} \in \mathbb{Z} \]

but since the LHS is positive it is in fact \(q+1\), thus

\[ \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + ... < \frac{1}{q+1} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + ... \]

\[ = \frac{1}{q+1} \cdot \frac{1}{1-\frac{1}{q+1}} = \frac{1}{e} \leq 1 \]

but this contradicts fact that 1 is the smallest positive integer.
(c) \(1 + a + \cdots + n = \frac{n(n+1)}{2}\).

Let \(n=1\), then \(1 = \frac{1 \cdot 2}{2}\) and so the above holds true.

Suppose that the above holds for \(n\). Then

\[(1+2+\cdots+n)(n+1) = \frac{n(n+1)}{2} + (n+1)\]

\[= (n+1)\left[\frac{n}{2} + 1\right] = (n+1)\left[\frac{n+2}{2}\right] = \frac{(n+1)(n+2)}{2} + (n+1)\]

Thus the formula is true for \(n+1\). Q.E.D.

(e) \(1^3 + 2^3 + \cdots + n^3 = (1+2+\cdots+n)^2\)

For \(n=1\), \(1^3 = 1^2\), so the formula is valid.

Suppose the formula is true for \(n\). Then

\[\left[1^3 + \cdots + n^3\right] + (n+1)^3 = \left(1+2+\cdots+n\right)^2 + (n+1)^2\]

\[= \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^2\] \hspace{1cm} \text{by (a)},

\[= (n+1)^2 \left[\frac{n^2}{4} + (n+1)\right]\]

\[= \frac{(n+1)^2}{4} \left[n^2 + 2n + 1\right]\]

\[= \left[\frac{(n+1)(n+2)}{2}\right]^2\]

\[= (1+2+\cdots+n+1)^2\] \hspace{1cm} \text{by (a)},

and so the formula is true for \(n+1\). Q.E.D.
\[ A2 \] (f) \quad S = \left\{ \frac{2}{3} x \mid 1 \leq x \leq 4 \right\}.

Suppose that \( x \geq 4 \); then \( x \in S \iff x + x - 4 \leq 6 \iff x \leq 5 \). Thus \( [4, 5] \subset S \).

Suppose that \( 0 \leq x \leq 4 \); then \( x \in S \iff x - (x - 4) \leq 6 \iff 4 \leq 6 \), which is always true. Thus \( [0, 4] \subset S \).

Suppose that \( x \leq 0 \); then \( x \in S \iff -x - (x - 4) \leq 6 \iff -2x \leq 2 \iff x \geq -1 \). Thus \( [-1, 0] \subset S \).

No together we have that: \( [-1, 0] \cup [0, 4] \cup [4, 5] = S \).

That is \( S = [-1, 5] \). \( \sup_S x = 5 \), \( \inf_S x = -1 \).

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\[ \mathbb{P}(S) = \frac{2}{3} \sum_{k=1}^{\infty} \frac{1}{k} = \frac{2}{3} \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \right) \Rightarrow \sup_S x = 5 \] \( \inf_S x = 1 \) (by inspection).

Verify that \( \sup_S x = 5 \). It suffices to show that \( S \) is an upper bound for \( S \), since \( 5 \) is in \( S \), and so there can be no smaller upper bound. Let \( n \) be a positive integer. Then \( n \geq 1 \), and so \( 1 \leq n \implies 4 \leq 4n \implies \frac{4}{n} \leq 4 \). Thus \( 1 + \frac{4}{n} \leq 5 \). Thus every \( x \) of \( S \) is \( \leq 5 \).

Verify that \( \inf_S x = 1 \). First we show that \( 1 \) is a lower bound. Let \( n \in \mathbb{N} \). Then \( n > 0 \) and so \( \frac{4}{n} > 0 \). Thus \( 1 + \frac{4}{n} > 1 \). Now we show that \( 1 \) is the greatest lower bound. Suppose not. Then \( \exists e > 0 \) such that \( 1 + e \not\leq 1 + \frac{4}{n} \) for all \( n \in \mathbb{N} \).

But this implies that \( \frac{4}{n} \geq e \), that is, \( n \leq \frac{4}{e} \), which contradicts the archimedean property of the reals. Hence \( 1 = \inf_S \).
a) Show that $\frac{1}{2} (a+b+|a-b|) = \max (a, b)$.

**Case i:** Suppose that $a \geq b$. Then
\[
\frac{1}{2} (a+b+|a-b|) = \frac{1}{2} (a+b+(a-b)) = \frac{1}{2} (2a) = a = \max (a, b).
\]

**Case ii:** Suppose that $b \geq a$. Then
\[
\frac{1}{2} (a+b+|a-b|) = \frac{1}{2} (a+b+(b-a)) = \frac{1}{2} (2b) = b = \max (a, b).
\]

b) Claim: $\frac{1}{2} (a+b-|a-b|) = \min (a, b)$.

**Case i:** $a \geq b$. Let $\text{LHS} = \frac{1}{2} (a+b-(a-b)) = b = \min (a, b)$

**Case ii:** $a \leq b$. Let $\text{LHS} = \frac{1}{2} (a+b+(b-a)) = a = \min (a, b)$.

1.7B b) Show that $(1-x)^n \leq 1-nx + \frac{n(n-1)}{2} x^2$, $0 \leq x < 1$.

For $n=1$, $1-x = 1-x$, so the above is true.

Suppose the above is true for $n$. Then
\[
(1-x)^{n+1} = (1-x)^n (1-x) \leq (1-nx + \frac{n(n-1)}{2} x^2) (1-x)
\]

because $1-x > 0$, thus $x < 1$.

\[
= 1-nx-x+n x^2 + \frac{n(n-1)}{2} x^2 - \frac{n(n-1)}{2} x^3
\]

\[
= 1-(n+1)x + \left(1 + \frac{n-1}{2}\right)n x^2 - P, \text{ where } P = \frac{n(n-1)}{2} x^3
\]

\[
= 1-(n+1)x + \frac{(n+1)n}{2} x^2 - P
\]

\[
\leq 1 - (n+1)x + \frac{(n+1)(n+1)}{2} x^2, \text{ since } P \geq 0. \text{ (because } x \geq 0)\]

Thus, the inequality is valid for $n+1$. QED

1.7B 7) Show that every nonempty set $S$ bounded below has an infimum.

Let $S$ be a nonempty subset of $\mathbb{R}$, and let $m$ be a lower bound for $S$. Let $T = \frac{3}{2} - x$ if $x \in S \subseteq \mathbb{R}$, and let $M = m$.
Then \( M \) is an upper bound for \( T \), for if \( x \in S \) then \( x \geq m \implies -x \leq -m \implies -x \leq M \). Thus \( T \) is a nonempty subset of \( \mathbb{R} \) bounded above and so by Thm 1.5.6, \( T \) has a supremum \( M_0 \). Let \( m_0 = -M_0 \). We claim that \( m_0 = \inf_S x \). There are two things to check.

(i) \( m_0 \) is a lower bound for \( S \), for if \( x \in S \) then \(-x \in T\)
and so \(-x \leq M_0 \), that is \( x \geq -M_0 = m_0 \).

(ii) If \( m \) is any lower bound for \( S \), then \(-m \) is an upper bound for \( T \) and so \(-m \geq M_0 \). Thus \( m \leq -M_0 = m_0 \). That is \( m_0 \) is the greatest lower bound for \( S \).

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8. Show that if \( S \) is bounded below and \( m_0 = \inf_S x \), then for each \( y > m_0 \) \( \exists x \in S \) for which \( y > x \geq m_0 \).

Let \( S \), suppose the statement is false. Then \( \exists y \) such that \( y > m_0 \) and such that the interval \([m_0, y)\) contains no \( x \in S \). This implies that \( y \) is a lower bound for \( S \) (since \( m_0 \) already is a lower bound). But \( y > m_0 \), contradicting the fact that \( m_0 \) is the greatest lower bound for \( S \). Thus, the statement must be true.