ALGEBRAIC SYSTEMS
Exam 2
October 26, 2012

The point value of each problem is given in the margin. Total = 80 points.

(6) 1. Find the cardinalities of the following sets, that is, the number of elements in the set.
   a) \(|\mathbb{Z}_{20}| = 20\)

   b) \(|\mathbb{U}_{20}| = \phi(20) = \phi(2^2 \cdot 5) = \phi(2^2) \phi(5)\) (the group of units in \(\mathbb{Z}_{20}\))

   \[= 2 \cdot 4 = 8\]

   c) \(|M_{2,2}(\mathbb{Z}_3)| = 3^4 = 81\) (the set of 2 by 2 matrices over \(\mathbb{Z}_3\))

(9) 2. a) Define what it means for an element \(a\) of a ring \(R\) to be a zero divisor.

   \[\text{\(a\) is a zero divisor if there exists a nonzero } b \in R \text{ such that } ab = 0 \text{ or } ba = 0. \quad \text{(Also, } a \neq 0.\text{)}\]

b) Define what it means for a ring \(R\) to be an integral domain.

   \[R \text{ is an integral domain if:}\]
   \[\begin{align*}
   \text{i) } & \text{\(R\) is commutative: } ab = ba \quad (\forall a, b \in R) \\
   \text{ii) } & \text{\(R\) has a unity element} \\
   \text{iii) } & \text{\(R\) has no zero divisors.}
   \end{align*}\]

c) State a beautiful relationship (equation) between 0, 1, e, \(\pi\) and \(i\).

   \[e^{\pi i} + 1 = 0\]

(2) 3. a) Express the congruence class \([7]_{12}\) in \(\mathbb{Z}_{12}\) two different ways:

   \[[7]_{12} = \{x \in \mathbb{Z} : x \equiv 7 \pmod{12}\}\]

   \[[7]_{12} = \{7 + k \cdot 12 : k \in \mathbb{Z}\}.\]

b) Find all zero divisors in \(\mathbb{Z}_{12}\). (No work needs to be shown.)

   \[\gcd(a, 12) > 1 \quad \{2, 3, 4, 6, 8, 9, 10\}\]

   \[a \neq 0\]

(2) c) Find all units in \(\mathbb{Z}_{12}\). (No work needs to be shown.)

   \[\gcd(a, 12) = 1 \quad \{1, 5, 7, 11\}\]
(3) 4. a) State Fermat’s Little Theorem, dealing with exponentiation \( \text{mod} \ p \).
   (Include all hypotheses, including what type of number \( p \) is.)

   If \( p \) is a prime and \( a \in \mathbb{Z} \) with \( p \nmid a \), then

   \[ a^{p-1} \equiv 1 \quad (\text{mod} \ p). \]

(5) 5. Let \( z = -\sqrt{3} + i \in \mathbb{C} \).
   (2) a) Plot \( z \) on a graph.

   \[ -\sqrt{3} + i \quad \Theta \quad \text{as} \quad \overrightarrow{30} \]
   \[ -\sqrt{3} \]

   b) Find \( |z| \), and the polar angle of \( z \).

   \[ |z| = \sqrt{3^2 + 1^2} = 2 \]
   \[ \Theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6} \]

   c) Give the exponential polar form of \( z \).

   \[ z = r e^{i\Theta} = 2 e^{i \frac{5\pi}{6}} \]

(4) d) Find \( z^6 \). Express your final answer in standard form \( a + bi \) with no trig functions.

   \[ z^6 = (2 e^{i \frac{5\pi}{6}})^6 = 2^6 e^{i 5\pi} = 2^6 (-1) = -64. \]

on unit circle with polar angle \( 5\pi \), so \(-1\)
6. Find \((-1)^{1/5}\), the set of all fifth roots of \(-1\) in \(\mathbb{C}\). Express your answers in polar form or exponential polar form.

\[-1 = e^{i \left( \frac{\pi + 2\pi k}{5} \right)}, \quad k \in \mathbb{Z}\]

\[-1 \frac{1}{5} = e^{i \left( \frac{\pi + 2\pi k}{5} \cdot \frac{1}{5} \right)} = e^{i \left( \frac{\pi}{5} + \frac{2\pi k}{5} \right)}, \quad k = 0, 1, 2, 3, 4\]

(8) 7. Short answer.

a) Give an example of an integral domain that is not a field and not equal to \(\mathbb{Z}\). \(\mathbb{R}[x]\)

b) Give an example of three different fields \(R, S, T\) with \(R \subset S \subset T\). \(\emptyset \subset \mathbb{R} \subset \mathbb{C}\)

c) Give an example of a noncommutative ring with infinitely many elements. \(M_{2,2}(\mathbb{R})\)

d) Give an example of a ring \(R\) and a nonzero element \(a\) of \(R\) such that \(a\) is neither a zero divisor nor a unit. Let \(R = \mathbb{Z}\), \(a = 2\)

(12) 8. Indicate whether the following sets are closed under addition (C.A.), closed under multiplication (C.M.), Rings (Ring), Commutative rings (C.Ring), Rings with Unity (R.U.), Integral Domains (Domain), Fields (Field). Circle all correct answers on each problem.

a) \(\mathbb{Z}_8\)  C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

b) \(\mathbb{R}[x]\)  C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

c) \(\{0, 1, 4, 8\}\) in \(\mathbb{Z}_{12}\)  C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

d) \(\left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{R} \right\}\)  C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

\(\text{If } a \neq 0 \text{ then } a^{-1} = \begin{bmatrix} a^{-1} & 0 \\ 0 & a^{-1} \end{bmatrix}\)

e) \(3\mathbb{Z} = \{3n : n \in \mathbb{Z}\}\)  C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

f) \(\{f(x) \in \mathbb{Z}[x] : \deg(f(x)) \geq 2\}\)  C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

Fail, \(x^2 + (-x^2) = 0\), \(0 \notin S\)
(4) 9. In \( \mathbb{Z}_3[x] \) find \((x + 1)^3\). Simplify your answer.
\[
(x + 1)^3 = x^3 + 3x^2 + 3x + 1^3
= x^3 + 0 + 0 + 1^3
= x^3 + 1
\]

(since \( \mathbb{Z}_3 \) is commutative)

since \( 3 \equiv 0 \) in \( \mathbb{Z}_3 \)

(5) 10. Prove that if \( m \) is a composite positive integer, then \( \mathbb{Z}_m \) is not an integral domain.

Suppose \( m \) is composite. Then \( m = \alpha \cdot \beta \) for some integers \( \alpha, \beta \) with \( 1 < \alpha < m, 1 < \beta < m \). Thus \([\alpha]_m \not\equiv [0]_m\), \([\beta]_m \not\equiv [0]_m\), but
\[
[\alpha]_m \cdot [\beta]_m = [\alpha \beta]_m = [m]_m = [0]_m
\]
that is \([\alpha]_m, [\beta]_m \) are zero divisors. Therefore \( \mathbb{Z}_m \) is not an integral domain.

(6) 11. Prove the cancellation law for \( \mathbb{Z}_m \): If \( \gcd (a, m) = 1 \) and \( ax \equiv ay \) (mod \( m \)), then \( x \equiv y \) (mod \( m \)). (You may assume appropriate lemmas.)

Suppose \( \gcd (a, m) = 1 \) and \( ax \equiv ay \) (mod \( m \)).

Then \( a \) has a mult. inverse \( a^{-1} \) (mod \( m \)) (since \( \gcd (a, m) = 1 \)).

Thus, by *, \( a^{-1}(ax) \equiv a^{-1}(ay) \) (mod \( m \))
\[
\Rightarrow (a^{-1}a)x \equiv (a^{-1}a)y \quad \text{(mod \( m \))}, \ by \ associ. \ law
\Rightarrow 1x \equiv 1y \quad \text{(mod \( m \))}, \ \text{def. \ of \ mult \ inv.}
\Rightarrow x \equiv y \quad \text{(mod \( m \))}, \ 1 \ is \ mult. \ identity.