ALGEBRAIC SYSTEMS
Exam 3
December 2, 2011
The point value of each problem is given in the margin. Total = 80 points.

(9) 1. Indicate whether the given polynomial is reducible or irreducible over the given field and explain why. You do not need to factor the polynomial.
   a) $x^5 + x + 1$ over $\mathbb{Z}_3$.
   b) $x^5 + 4x - 24$ over $\mathbb{R}$.
   c) $x^3 - x + 2$ over $\mathbb{Q}$.

(8) 2. Prove the following part of the factor theorem: Let $F$ be a field, $f(x) \in F[x]$ and $a \in F$. If $a$ is a zero of $f(x)$ then $(x - a)$ is a factor of $f(x)$.
3. Completely factor the polynomial $x^3 - 5x + 2$ over $\mathbb{Q}$ and over $\mathbb{R}$. (I recommend not using Cardano’s method!)

4. a) Suppose that $f(x)$ is a fifth degree polynomial over $\mathbb{Q}$, with no zero in $\mathbb{Q}$. What are the possible factorizations of $f(x)$ over $\mathbb{Q}$ into a product of irreducibles?

b) State the Fundamental Theorem of Algebra.

c) Let $F$ be a given field and $f(x) \in F[x]$. Define (precisely) what it means for $f(x)$ to be reducible over $F$:

d) Give examples of three different cyclic groups of order 4. Make sure to specify the binary operation.
5. Prove the conjugate zero theorem: If \( f(x) \in \mathbb{R}[x] \) and \( z \in \mathbb{C} \) is a zero of \( f(x) \) then \( \overline{z} \) is also a zero of \( f(x) \).

6. Let \( G \) be the group \((\mathbb{Z}_{14}, +)\).
   a) What is the identity element?

   b) What is the inverse of \( 8 \) in \( G \)? Explain.

   c) What is the order of \( 7 \) in \( G \)? Explain.

7. Determine whether the following sets are groups under the given operation. If it is a group just say so, and give the identity element. If it is not a group, state one axiom that fails.
   a) The set of even integers \( E = \{2n : n \in \mathbb{Z}\} \) under multiplication.

   b) The set of nonnegative rational numbers \( \{x : x \in \mathbb{Q}, x \geq 0\} \) under addition.

   c) The set of positive real numbers under multiplication.
8. Let $U_{15}$ be the multiplicative group of units (mod 15).
   a) List the elements of $U_{15}$.
   
   b) What is the inverse of 4 in $U_{15}$?
   
   c) Find the subgroup $\langle 2 \rangle$, generated by 2.
   
   d) Find the order of 2 in $U_{15}$.

9. Let $C_{10} = \langle a \rangle$ denote a cyclic group of order 10 under multiplication. (For example, $C_{10}$ could be $U_{11}$ or $\langle \omega \rangle$ where $\omega = e^{2\pi i/11}$.)
   a) Find all subgroups of $C_{10}$.
   
   b) Find all generators for $C_{10}$.

10. Find a monic polynomial $f(x)$ of degree 5 over the rationals such that $f(0) = f'(0) = 0$, $f(2 - 3i) = 0$, $f(1) = 20$. (Your final answer can be expressed as a product, but the factors should all have rational coefficients.)