ALGEBRAIC SYSTEMS
Exam 2
October 31, 2011

The point value of each problem is given in the margin. Total = 80 points.

(6) 1. Define the multiplication law for \( \mathbb{Z}_m \) and prove that it is well defined.
(You may assume properties of congruences.)
For any \([a]_m, [b]_m \in \mathbb{Z}_m\), \([a]_m[b]_m = \ldots\).

(6) 2. Let \( R \) be a ring with unity 1, and \( a \in R \). Complete the following definitions.
   a) \( a \) is a zero divisor if

   b) \( a \) is a unit if

(2) 3. a) Express the congruence class \([2]_{10}\) in \( \mathbb{Z}_{10} \) two different ways:
\[ [2]_{10} = \{ x \in \mathbb{Z} : x \equiv \ldots \pmod{\ldots} \} \]
\[ [2]_{10} = \{ \ldots + k \ldots : k \in \mathbb{Z} \}. \]

(2) b) Find all zero divisors in \( \mathbb{Z}_{10} \). (No work needs to be shown.)

(2) b) Find all units in \( \mathbb{Z}_{10} \). (No work needs to be shown.)
4. a) State Euler’s Theorem, dealing with exponentiation $\mod \ m$.
   (Include all hypotheses.)

6. b) Use Euler’s Theorem to evaluate $7^{82} \mod 60$

5. Let $z = 2 - 2i \in \mathbb{C}$.
   (2) a) Find $|z|$, and the polar angle of $z$.

   (2) b) Give the exponential polar form of $z$.

   (4) c) Find $z^6$. Express your final answer in standard form $a + bi$ with no trig functions.

6. Find $i^{1/6}$, the set of all sixth roots of $i$ in $\mathbb{C}$. Express your answers in polar form or exponential polar form.
(8) 7. Short answer.
   a) Give an example of an integral domain that is not a field and not equal to \( \mathbb{Z} \).

   b) Give an example of a finite field, that is, a field with finitely many elements.

   c) Give an example of a noncommutative ring with 16 elements.

   d) Give an example of a ring that is not a ring with unity.

(12) 8. Indicate whether the following sets are closed under addition (C.A.), closed under multiplication (C.M.), Rings (Ring), Commutative rings (C.Ring), Rings with Unity (R.U.), Integral Domains (Domain), Fields (Field). Circle all correct answers on each problem.

   a) \( \mathbb{Z}_6 \)  C.A. C.M. Ring C.Ring R.U. Domain Field

   b) \( \mathbb{Q}[x] \)  C.A. C.M. Ring C.Ring R.U. Domain Field

   c) \( \{0, 3, 6, 9, 12\} \) in \( \mathbb{Z}_{15} \)  C.A. C.M. Ring C.Ring R.U. Domain Field

   d) \( \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{Z} \right\} \)  C.A. C.M. Ring C.Ring R.U. Domain Field

   e) \( M_{2,2}(\mathbb{Z}_5) \)  C.A. C.M. Ring C.Ring R.U. Domain Field

   f) \( \{ f(x) \in \mathbb{Z}[x] : f(x) \text{ has only even powers of } x, \text{ including } x^0. \} \)  C.A. C.M. Ring C.Ring R.U. Domain Field

(6) 9. Prove that \( \overline{z} w = \overline{z} \cdot \overline{w} \) for any two complex numbers \( z = a + bi, w = c + di \).
(6) 10. Prove that if $x$ is a unit in a ring $R$ then $x$ is not a zero divisor.

(4) 11. a) In $\mathbb{Z}_9[x]$ find $(1 + 3x)^3$.

(2) b) Find the multiplicative inverse of $1 + 3x$ in $\mathbb{Z}_9[x]$. (Hint: Use part (a).)