ALGEBRAIC SYSTEMS
Exam 1
October 2, 2009

The point value of each problem is given in the margin. You may use a calculator to check your answers, but show all work here. Total = 80 points.

(10) 1. Complete the statement of the cancellation law for multiplication and then prove it, justifying each step of the proof. (You may assume the property: \( x - y = 0 \) implies \( x = y \). Do not use fractions.)

Let \( a, x, y \in \mathbb{Z} \) with \( ax = ay \) and \( a \neq 0 \). Then \( \frac{x}{a} = \frac{y}{a} \).

Proof.

\[
\begin{align*}
ax &= ay \\
\Rightarrow ax - ay &= ay - ay \\
\Rightarrow a(x - y) &= 0 \\
\Rightarrow a = 0 \quad \text{or} \quad (x - y) = 0 \\
\Rightarrow x - y &= 0, \quad \text{since } a \neq 0 \\
\Rightarrow x &= y \quad \text{by property assumed in the directions.}
\end{align*}
\]

(10) 2. Use the Euclidean Algorithm to find the greatest common divisor \( d \) of 209 and 187 and find integers \( x, y \) such that \( 209x + 187y = d \). (You can use any method you like.)

\[
\begin{array}{c|c|c|c|c|c}
209x + 187y & 209 & 187 & 22 & 11 \\
\hline
22 & 187 & 22 & 11 & \text{gcd}(209,187) \\
\hline
x & 1 & 0 & 1 & -8 \\
\hline
y & 0 & 1 & -1 & 9 \\
\end{array}
\]

\[22 \div 187 \overset{8}{176} \overset{11}{11}\]

Thus \( \text{gcd}(209,187) = 11 \) and we have

\[209(-8) + 187(9) = 11\]
(10) 3. Let \( \{F_n\}_n^{\infty} = \{1, 1, 2, 3, 5, 8, 13, 21, \ldots \} \) be the sequence of Fibonacci numbers and 
\( S_n = \sum_{k=1}^{n} F_k = F_1 + F_2 + \cdots + F_n \). Prove by induction that 
\( S_n = F_{n+2} - 1 \) for all \( n \in \mathbb{N} \). (Recall: 
\( F_{n+1} = F_n + F_{n-1} \) for \( n \geq 2 \).

**Base case:** When \( n = 1 \), \( S_1 = F_1 = 1 \), while \( F_{n+2} - 1 = F_3 - 1 = 2 - 1 = 1 \).

Thus \( S_1 = F_3 - 1 \), so the statement is true.

**Induction Assump:** Suppose the statement is true for a given \( n \), that is, 
\( S_n = F_{n+2} - 1 \).

Now consider the case \( n+1 \):

\[
S_{n+1} = (F_1 + F_2 + \cdots + F_n) + F_{n+1} = S_n + F_{n+1}
\]

\[
= F_{n+2} - 1 + F_{n+1} \quad \text{by induc. assump.}
\]

\[
= (F_{n+2} + F_{n+1}) - 1
\]

\[
= F_{n+3} - 1 \quad \text{by def. of Fib. seq.}
\]

\[
= F_{(n+1)+2} - 1 \quad \text{Q.E.D.}
\]

(10) 4. Use properties of congruences to compute the following numbers modulo 7. Find the least residues (i.e. the least nonnegative values). Make the arithmetic as easy as possible.

(a) \( 7030 \cdot 78 - 1403 \) (mod 7)

\[
7030 \equiv 3 \overset{\text{mod } 7}{\equiv} 2 \quad (\text{mod 7}), \quad 1403 \equiv 3 \quad (\text{mod 7})
\]

\[
78 \equiv 1 \quad (\text{mod 7})
\]

\[
\equiv 2 \cdot 1 - 3 \quad (\text{mod 7})
\]

\[
\equiv -1 \quad (\text{mod 7})
\]

\[
\equiv [6] \quad (\text{mod 7})
\]

(b) \( (77)^8 + (69)^5 \) (mod 7)

\[
77 \equiv 0 \quad (\text{mod 7}), \quad 69 \equiv -1 \quad (\text{mod 7})
\]

\[
\equiv 0^8 + (-1)^5 \quad (\text{mod 7})
\]

\[
\equiv 0 - 1 \quad (\text{mod 7})
\]

\[
\equiv [6] \quad (\text{mod 7})
\]
5. Find \(7^{121} \pmod{48}\).

\[
7^2 = 49 \equiv 1 \pmod{48}.
\]

Thus

\[
7^{121} \equiv 7^{2 \cdot 60 + 1} \equiv (7^2)^{60} \cdot 7 \equiv 1^{60} \cdot 7 \equiv 7 \pmod{48}
\]

6. Prove that if \(a \equiv b \pmod{m}\) and \(c \equiv d \pmod{m}\) then \(ac \equiv bd \pmod{m}\).

\[
\begin{align*}
q & \equiv b \pmod{m} \Rightarrow a = b + mx \text{ for some } x \in \mathbb{Z}.
\end{align*}
\]

\[
\begin{align*}
c & \equiv d \pmod{m} \Rightarrow c = d + my \text{ for some } y \in \mathbb{Z}.
\end{align*}
\]

Thus

\[
ac = (b + mx)(d + my) = bd + mxd + bmy + mxmy
\]

\[
\Rightarrow ac = bd + m(dx + by + mxy) = bd + m \text{ (integer)}
\]

\[
\Rightarrow ac \equiv bd \pmod{m}.
\]

7. Prove that there exist infinitely many primes.

**Proof by contradiction:** Suppose there are finitely many primes \(p_1, p_2, \ldots, p_k\). Let

\[
N = p_1 p_2 \cdots p_k + 1.
\]

Since \(N \in \mathbb{N}\) and \(N > 1\), it has at least one prime factor (by FTA). Call it \(p_i\), where \(1 \leq i \leq k\). Then we have

\[
p_i \mid N \text{ and } p_i \mid (p_1 p_2 \cdots p_k).
\]

By a basic divisibility property it follows that

\[
p_i \mid (N - p_1 p_2 \cdots p_k),
\]

that is, \(p_i \mid 1\), a contradiction.

Therefore, there exist infinitely many primes.
8. Complete the statements of the following theorems.
a) The division algorithm: Let \( a, b \in \mathbb{Z} \). Then there exist integers \( q, r \) such that...
\[
a = q \cdot b + r \quad \text{with} \quad 0 \leq r < b.
\]
b) GCDLC Theorem: If \( a, b \in \mathbb{Z} \), not both zero, and \( d = \gcd(a, b) \), then... there exist integers \( x, y \) such that \( ax + by = d \).

9. a) Use a calculator to find the continued fraction expansion of \( 1/\pi \). (Go down at least five places, so you can do part (b).)

\[
\frac{1}{\pi} = 0.31830988... = 0 + \frac{1}{3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \ldots}}}}}
\]

b) Find the first five convergents of the continued fraction in part (a), and the best rational approximation to \( 1/\pi \) having a denominator less than 400.

<table>
<thead>
<tr>
<th>0</th>
<th>3</th>
<th>7</th>
<th>15</th>
<th>1</th>
<th>292</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>17</td>
<td>106</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>22</td>
<td>33</td>
</tr>
</tbody>
</table>

Best approximation is
\[
\frac{113}{355} = 0.3183098592...
\]

113
355

0, \( \frac{1}{3} \), \( \frac{2}{7} \), \( \frac{106}{333} \), \( \frac{113}{355} \)

c) How close is your approximation in (b) to \( 1/\pi \) (Give number of decimal places of accuracy.)

\[
\frac{1}{\pi} = 0.3183098862...
\]

\[
\frac{113}{355} = 0.3183098592...
\]

7 place accuracy!