ALGEBRAIC SYSTEMS
Final Exam
December 18, 2008

The point value of each problem is given in the margin. Total = 160 points.

(15) 1. State the following definitions. Let $a, b, m$ be positive integers.

a) $a$ divides $b$:

b) $a \equiv b \pmod{m}$:

c) The residue class $[a]_m$:

d) Unit (in a ring with unity):

e) Integral Domain:

(12) 2. Prove by induction that for any positive integer $n$, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. 
3. Calculate the following values using properties of congruences.

a) \( 301^3 \cdot 149 + (67265)^{139} \pmod{5} \)

b) \( 2^{163} \pmod{17} \)

c) \( 4^{-1} \pmod{15} \)

4. Prove that \( \sqrt{p} \) is irrational for any prime \( p \).
5. Solve the equation \( z^6 = i \) in \( \mathbb{C} \), that is find \( i^{1/6} \).

6. a) State the quadratic formula for \( \mathbb{C} \). (State it carefully as a theorem with hypotheses and conclusion.)

b) Prove the theorem you stated in part (a).

7. a) Demonstrate that 8 is a zero divisor in \( \mathbb{Z}_{10} \) using the definition of zero divisor.

b) Find \( |U_{200}| \), where \( U_{200} \) is the multiplicative group of units \( \pmod{200} \).
8. Indicate whether the following sets are closed under addition (C.A.), closed under multiplication (C.M.), Additive Groups (A.Group), Rings (Ring), Integral Domains (Domain), Fields (Field). Circle all correct answers on each problem.

a) $\mathbb{Z}_7$  
   C.A. C.M. A.Group Ring Domain Field

b) $\mathbb{Q}$  
   C.A. C.M. A.Group Ring Domain Field

c) $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{Z} \right\}$  
   C.A. C.M. A.Group Ring Domain Field

d) $\mathbb{E}$, the set of even integers  
   C.A. C.M. A.Group Ring Domain Field

e) $\{a + bi : a, b \in \mathbb{Z}\} \ (i = \sqrt{-1})$  
   C.A. C.M. A.Group Ring Domain Field

f) $\mathbb{Z}_2[x]$  
   C.A. C.M. A.Group Ring Domain Field

9. Let $f(x) = x^5 + 16x + 2$. Describe the factorization of $f(x)$ over the following fields. Justify your answers. (That is, say it is irreducible, or a product of two linear and one irreducible cubic, etc. You don’t need to find the factors.)

a) $\mathbb{C}$

b) $\mathbb{R}$. (Note that $f'(x) = 5x^4 + 16 > 0$ for all $x$ so $f(x)$ is increasing.)

c) $\mathbb{Q}$. (You may describe more than one possible answer, based on methods you’ve learned so far, if you can’t make a definitive conclusion.)
10. Short Answer.
a) State a beautiful relationship between $\pi, e, 1$ and $i$. (There is more than one correct answer.)

b) Give an example of a cyclic group of order 12 under addition.

c) Give an example of a cyclic group of order 12 under multiplication.

d) How many subgroups does a cyclic group of order 12 have? (explain)

e) Give an example of a nonabelian group.

11. Let $\alpha \in S_7$ be the permutation

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 2 & 1 & 7 & 6 \end{pmatrix},$$

a) Express $\alpha$ as a product of disjoint cycles.

b) Find $\alpha^{-1}$ (in standard form or cycle form).

c) Let $\beta = (1,2)$. Find $\beta \alpha \beta$

d) Find the order of $\alpha$. 
12. Prove that if $G$ is a finite group of order $n$ and $x \in G$ then $\text{ord}(x)$ is a divisor of $n$.

13. Let $G$ be the additive group $(\mathbb{Z}_{10}, +)$.
   a) Find $\text{ord}(4) =$
   b) Find the inverse of 3 (with respect to the group operation).
   c) What is the identity element?
   d) Is $G$ a cyclic group? (Explain)

14. Let $D_6 = \langle \sigma, \tau \rangle$ be the dihedral group of symmetries of a hexagon, with $\sigma = (1, 2, 3, 4, 5, 6)$ being rotation clockwise 60° and $\tau = (2, 6)(3, 5)$ being a reflection. (The vertices are labeled in a clockwise direction.)
   a) Draw a picture of the hexagon with labeled vertices and indicate the axis of reflection for $\tau$.
   b) Find $\sigma \tau$ (in cycle notation) and indicate what symmetry of the square it is. (You can use your picture if you like to describe the answer.)
   c) Find $\text{ord}(\sigma) =$ and $\text{ord}(\tau) =$
   d) Find the order of $D_6$: $|D_6| =$
   e) Give a subgroup of $D_6$ of order 4. (Indicate the elements in terms of $\sigma$ and $\tau$.)