ALGEBRAIC SYSTEMS  
Exam 3  
December 3, 2008  
The point value of each problem is given in the margin. Total = 80 points.

(12) 1. Indicate whether the given polynomial is reducible or irreducible over the given field and explain why.

a) $X^3 - 7$ over $\mathbb{Q}$.

b) $1 + x + x^5$ over $\mathbb{Z}_3$.

c) $x^7 + x + a$ over $\mathbb{R}$, where $a$ is a nonzero real number.

(8) 2. Prove the following part of the factor theorem: Let $F$ be a field and $f(x) \in F[x]$. If $a \in F$ is a zero of $f(x)$ then $(x-a)$ is a factor of $f(x)$.
3. Completely factor the polynomial \( x^3 + 2x - 3 \) over \( \mathbb{R} \) and then over \( \mathbb{C} \). (Hint: 1 is obviously a zero, so don’t use Cardano’s method!)

4. Short answer.
   a) Suppose that \( f(x) \) is a fourth degree polynomial over \( \mathbb{R} \) with no zero in \( \mathbb{R} \). What are the possible factorizations of \( f(x) \) into a product of irreducible polynomials over \( \mathbb{R} \)?
   
   b) What are the possible factorizations of a fourth degree polynomial \( f(x) \) over \( \mathbb{Q} \) into irreducible polynomials over \( \mathbb{C} \).
   
   c) Find a polynomial \( f(x) \) of degree 4 over \( \mathbb{R} \) such that \( f(2) = f'(2) = 0 \) and \( f(i) = 0 \).
   
   d) State the Fundamental Theorem of Algebra.
5. Prove the conjugate zero theorem: If \( f(x) \in \mathbb{R}[x] \) has a zero \( z \in \mathbb{C} \), then \( \overline{z} \) is also a zero of \( f(x) \).

6. State the four axioms required for a set \( G \) to be a group under a binary operation \( \ast \).

7. Determine whether the following sets are groups under the given operation. If it is a group just say so, and give the identity element. If it is not a group, state one axiom that fails.
   a) The set of even integers \( \mathbb{E} = \{2n : n \in \mathbb{Z}\} \) under addition.
   b) The set of positive real numbers \( \mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\} \) under addition.
   c) The set of positive real numbers \( \mathbb{R}^+ \) under multiplication.
   d) The set of nonzero elements of \( \mathbb{Z}_{25} \) under multiplication.
(8) 5) Let $U_7$ be the multiplicative group of units (mod 7).

a) What is the order of $U_7$?

b) Find the subgroup $< 3 >$

c) Give the order of 2 in $U_7$ (and show why). $: \text{ord}(2) =$

d) Is $U_7$ a cyclic group? Explain.

(6) 9) a) Prove that if $G$ is a group and $x, y \in G$, then $(xy)^{-1} = y^{-1}x^{-1}$.

(2) b) If $G$ is an abelian group and $x, y \in G$, what does $(xy^{-1}x^{-1})^{-1}$ simplify to? (explain)