ALGEBRAIC SYSTEMS

Exam 2
November 7, 2008

The point value of each problem is given in the margin. Total = 80 points.

6. 1. Find the quotient and remainder when \( x^4 + 3 \) is divided by \( 2x^3 + 4 \) in \( \mathbb{Z}_5[x] \). Then state the relationship between the four polynomials (as given in the division algorithm).

\[
\begin{align*}
2x^3 + 4 & \longdiv{x^4 + 3} \\
3x(2x^3 + 4) & \rightarrow \frac{x^4 + 2x}{-2x + 3} \\
\text{Quotient} &= 3x \\
\text{Remainder} &= 3x + 3 \quad \text{(or } -2x + 3) \\
\end{align*}
\]

\[
x^4 + 3 = 3x(2x^3 + 4) + 3x + 3
\]

8. a) Find the set of units \( U_{10} \) in \( \mathbb{Z}_{10} \). \( a \) is a unit \( \Leftrightarrow d(\mathbb{Z}_{10}) = 1 \\
U_{10} = \{1, 3, 7, 9\}
\]

b) Find all zero divisors in \( \mathbb{Z}_{10} \) and show explicitly (ab=0) why each is a zero divisor. (Remember 0 is not called a zero divisor.)

Zero divisors: \( \{2, 4, 5, 6, 8\} \)

2 \cdot 5 = 0
4 \cdot 5 = 0
6 \cdot 5 = 0
8 \cdot 5 = 0

4. Find \( \phi(880) \), the number of integers from 1 to 880 that are relatively prime to 880.

\[
\begin{align*}
880 &= 2^4 \cdot 5 \cdot 11 \\
\phi(880) &= \phi(2^4)\phi(5)\phi(11) = (2^4 - 2^3)(5-1)(11-1) \\
&= 8 \cdot 4 \cdot 10 = 320
\end{align*}
\]

(2) 4. Find the set of units in \( \mathbb{R}[x] \).

All non-zero constant polynomials.
5. a) State Euler’s Theorem. (Dealing with modular exponentiation \( \text{mod } m \).)

Let \( m \in \mathbb{Z} \), \( \text{if } q \in \mathbb{Z} \) and \( \gcd(a, m) = 1 \) then

\[
\alpha^\phi(m) \equiv 1 \pmod{m}
\]

b) Use Fermat’s Little Theorem or Euler’s Theorem to evaluate \( 17^{193} \pmod{97} \)

Note 97 is a prime since 2, 3, 5, 7 are not divisors. Also \( \gcd(17, 97) = 1 \)
Thus by FLT (or Euler)

\[
17^{96} \equiv 1 \pmod{97}
\]

Thus

\[
17^{193} = 17^{2 \cdot 96 + 1} = (17^{96})^2 \cdot 17 \equiv 1^2 \cdot 17 \pmod{97}
\]

\[
\equiv 17 \pmod{97}
\]

6. Let \( z = 2 - 2i \in \mathbb{C} \).

a) Find \( |z| \) and the exponential polar form of \( z \).

\[
|z| = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}
\]

\[
\Theta = -\pi/4 \quad \text{(or ) } \frac{7\pi}{4}
\]

\[
z = r e^{i\Theta} = 2\sqrt{2} \ e^{-i \frac{\pi}{4}}
\]

b) Find \( z^6 \). Express your final answer in standard form \( a + bi \) with no trig functions.

\[
z^6 = (2\sqrt{2})^6 \left(e^{-i \frac{\pi}{4}}\right)^6 = 2^6 \cdot 2^3 \cdot e^{-i \frac{3\pi}{2}} = 2^9 \cdot i
\]
7. Short answer.

a) What is the cardinality of the set of two-by-two matrices \( M_{2,2}(\mathbb{Z}_3) \)? Each entry has 3 choices, so \( |M_{2,2}(\mathbb{Z}_3)| = 3^4 \).

b) In order to verify that a subset of a given ring is also a ring, which axioms must be verified, and which are inherited? Must verify closed under +, closed under \( \cdot \), has 0 element and has additive inverses. Associative laws, commut. law for + and distrib. laws are inherited.

c) Give an example of three different fields \( F, G, H \) with \( F \subset G \subset H \).

\[ \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \]

d) Give an example of a ring without unity.

\[ LE = \{ \frac{1}{2} \alpha n : n \in \mathbb{Z} \} \]

8. Indicate whether the following sets are closed under addition (C.A.), closed under multiplication (C.M.), Rings (Ring), Commutative rings (C.Ring), Rings with Unity (R.U.), Integral Domains (Domain), Fields (Field). Circle all correct answers on each problem.

a) \( \mathbb{Z}_7 \)  \( \square \)  C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

b) \( M_{2,2}(\mathbb{C}) \)  \( \square \)  C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

c) \( \mathbb{Z}_4[x] \)  \( \square \)  C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

d) \( \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{Z} \)  \( \square \)  C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

\[ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a \cdot \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix} \]

\[ ab = ba \]

If \( ab = 0 \) then \( a = 0 \) or \( b = 0 \).

e) \{0, [3], [6] \} in \( \mathbb{Z}_9 \)  \( \square \)  C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

f) \[ \{4n : n \in \mathbb{Z} \} \]  \( \square \)  C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

9. Prove that if \( x \) is a unit in a ring with unity then \( x \) is not a zero divisor. (You may assume the property that \( a \cdot 0 = 0 \) for any \( a \).)

Suppose \( x \) is a unit in a ring \( R \) with unity \( 1 \). Let \( y \in R \) such that \( x \cdot y = 0 \). Then \( x^{-1} (x \cdot y) = x^{-1} \cdot 0 \) is well defined.

\[ (x \cdot x)^{-1} \cdot y = 0 \]

assoc. law

\[ 1 \cdot y = 0 \]

mult. inverse

\[ y = 0 \]

mult. identity

Therefore \( x \) is not a zero divisor.
10. Prove that for any complex numbers \( z = a + bi \) and \( w = c + di \) that \( \overline{zw} = \overline{z} \overline{w} \).

\[
\overline{zw} = \overline{(a+bi)(c+di)} = \overline{ac-bd + (bc+ad)i} = (ac-bd) - (bc+ad)i
\]

\[
\overline{z} \overline{w} = (a-bi)(c-di) = (ac-bd) - (bc+ad)i
\]
Comparing the right-hand sides we see that \( \overline{zw} = \overline{z} \overline{w} \).

11. Find all numbers of the form \( a2b6 \) divisible by 66, using divisibility properties.

\[ 66 = 6 \cdot 11 = 2 \cdot 3 \cdot 11 \]  
The last digit is even so we only need to test for divisibility by 3 and 11.

Test 11: \( a - 2 + b - a + 6 = b + 4 \Rightarrow 0, 11, 22, \ldots \) Thus \( b = 7 \).

Test 3: \[ a + 2 + 7 + a + 6 = 2a + 15 = 15, 18, 21, 24, 27, 30, 33, \ldots \]  
\[ \Rightarrow 2a = 0, 3, 6, 9, 12, 15, 18 \]

\[ \Rightarrow a = 0, 3, 6, 9 \]
Thus \( 2706, 32736, 62766, 92796 \)

12. Find all fifth roots of \( -1 \) in \( \mathbb{C} \). Express your answers in polar form or exponential polar form and plot the points on the unit circle.

Polar form: \( -1 = 1 \cdot e^{\pi i} = 1 \cdot e^{(\pi + 2\pi j)i} \quad j \in \mathbb{Z} \)

\[ -1^{\frac{1}{5}} = \sqrt[5]{1} \cdot e^{\left(\frac{\pi + 2\pi j}{5}\right)i} \]

\[ = e^{\left(\frac{\pi + 2\pi j}{5}\right)i} \quad j = 0, 1, 2, 3, 4 \]

Note the complex conjugate pairs:

\( e^{\frac{\pi}{5}i} = e^{\frac{6\pi}{5}i} \)

\( e^{\frac{2\pi}{5}i} = e^{\frac{7\pi}{5}i} \)

\[ \frac{\pi}{5} = \frac{180}{5} \times 72 = 36^\circ \]

\[ \frac{2\pi}{5} = \frac{3\pi}{5} = 108^\circ \]

\[ \frac{3\pi}{5} = \frac{180}{72} = 108^\circ \]

\[ \frac{4\pi}{5} = \frac{2\pi}{5} = 72^\circ \]

\[ \frac{5\pi}{5} = \frac{2\pi}{5} = 252^\circ \]

\[ \frac{6\pi}{5} = \frac{2\pi}{5} = 324^\circ \text{ or } -36^\circ \]