ALGEBRAIC SYSTEMS
Exam 2
November 7, 2008
The point value of each problem is given in the margin. Total = 80 points.

1. Find the quotient and remainder when \( x^4 + 3 \) is divided by \( 2x^3 + 4 \) in \( \mathbb{Z}_5[x] \). Then state the relationship between the four polynomials (as given in the division algorithm).

2. a) Find the set of units \( U_{10} \) in \( \mathbb{Z}_{10} \).

b) Find all zero divisors in \( \mathbb{Z}_{10} \) and show explicitly (ab=0) why each is a zero divisor. (Remember 0 is not called a zero divisor.)

3. Find \( \phi(880) \), the number of integers from 1 to 880 that are relatively prime to 880.

4. Find the set of units in \( \mathbb{R}[x] \).
5. a) State Euler’s Theorem. (Dealing with modular exponentiation \((\text{mod } m)\).)

b) Use Fermat’s Little Theorem or Euler’s Theorem to evaluate \(17^{193} \pmod{97}\)

6. Let \(z = 2 - 2i \in \mathbb{C}\).

a) Find \(|z|\) and the exponential polar form of \(z\).

b) Find \(z^6\). Express your final answer in standard form \(a + bi\) with no trig functions.
7. Short answer.
   a) What is the cardinality of the set of two-by-two matrices $M_{2,2}(Z_3)$.

   b) In order to verify that a subset of a given ring is also a ring, which axioms must be verified, and which are inherited?

   c) Give an example of three different fields $F, G, H$ with $F \subset G \subset H$.

   d) Give an example of a ring without unity.

8. Indicate whether the following sets are closed under addition (C.A.), closed under multiplication (C.M.), Rings (Ring), Commutative rings (C.Ring), Rings with Unity (R.U.), Integral Domains (Domain), Fields (Field). Circle all correct answers on each problem.

   a) $Z_7$  C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

   b) $M_{2,2}(C)$  C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

   c) $Z_4[x]$  C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

   d) $\left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in Z \right\}$  C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

   e) $\{[0], [3], [6]\}$ in $Z_9$  C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

   f) $\{4n : n \in Z\}$  C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

9. Prove that if $x$ is a unit in a ring with unity then $x$ is not a zero divisor. (You may assume the property that $a \cdot 0 = 0$ for any $a$.)
10. Prove that for any complex numbers \( z = a + bi \) and \( w = c + di \) that \( \overline{zw} = \overline{z} \cdot \overline{w} \).

11. Find all numbers of the form \( a2ba6 \) divisible by 66, using divisibility properties.

12. Find all fifth roots of \(-1\) in \( \mathbb{C} \). Express your answers in polar form or exponential polar form and plot the points on the unit circle.