ALGEBRAIC SYSTEMS
Exam 1
October 1, 2008

The point value of each problem is given in the margin. You may use a calculator to check your answers, but show all work here. Total = 80 points.

(10) 1. Prove the cancellation law for addition, justifying each step of the proof.
    Let \( a, x, y \in \mathbb{Z} \). If \( a + x = a + y \) then \( x = y \).

Proof.

(10) 2. Use the Euclidean Algorithm to find the greatest common divisor \( d \) of 182 and 143 and find integers \( x, y \) such that \( 182x + 143y = d \). (You can use any method you like.)
(10) 3. Let \( r \) be a real number, \( r \neq 1 \). Prove by induction that for any positive integer \( n \)

\[
1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1},
\]

that is, \( \sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1} \). (This is the familiar formula for the sum of a geometric series. You may know another proof, but for full credit you need to use induction.)

(10) 4. Use properties of congruences to compute the following numbers modulo 6. Find the least residues (i.e. the least nonnegative values). Make the arithmetic as easy as possible.

(a) \( 662 \cdot 601 - 1203 \pmod{6} \)

(b) \( 65^8 + 62^5 \pmod{6} \)
(5) 5. Find $3^{121} \pmod{28}$.

(9) 6. Prove that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then $ac \equiv bd \pmod{m}$.

(10) 7. Prove that $\sqrt{p}$ is irrational for any prime $p$. 
8. a) Find all integers that have multiplicative inverses (mod 18).

b) What is the multiplicative inverse of 5 (mod 18)?

9. a) Use a calculator to find the continued fraction expansion of 5.4321. (Go down at least five places, so you can do part (b).)

b) Find the best rational approximation to 5.4321 having a denominator less than 100.