ALGEBRAIC SYSTEMS
Exam 3
November 30, 2007
The point value of each problem is given in the margin. Total = 80 points.

(10) 1) Indicate whether the given polynomial is reducible or irreducible over the given ring. Circle the correct answer. No work needs to be shown.

a) \(3x^2 + 6\) over \(\mathbb{Q}\). reducible irreducible

b) \(x^5 + 2\) over \(\mathbb{Z}_3\). reducible irreducible

c) \(x^7 + x + 17\) over \(\mathbb{C}\). reducible irreducible

d) \(x^3 + x + 2\) over \(\mathbb{Q}\). reducible irreducible

e) \(x^4 + x^2 + 17\) over \(\mathbb{R}\). reducible irreducible

(10) 2. Prove the Rational Root Test: Let \(f(x) = a_nx^n + \cdots + a_0\) be a polynomial of degree \(n\) over \(\mathbb{Z}\) and \(\frac{r}{s}\) be a rational root of \(f(x)\) with \(r, s\) relatively prime integers. Then \(r|a_0\) and \(s|a_n\).
(Just prove one of the two divisibility conditions. The other follows in a similar manner.)
3. Completely factor the polynomial $x^4 + 1$ over $\mathbb{C}$ and over $\mathbb{R}$.

4. Determine whether the following sets are groups under the given operation. If not, state one axiom that fails.
   a) $\{\pm 1, \pm i\}$ under multiplication. (Here, $i = \sqrt{-1}$.)

   b) The set of powers of 2, $\{2^k : k \in \mathbb{Z}\} = \{..., \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, ...\}$, under multiplication.

   c) The set of powers of 2 under addition.

   d) The set of natural numbers under multiplication.

5. Let $U_9$ be the multiplicative group of units (mod 9).
   a) What is the order of $U_9$?

   b) Find the subgroup $< 2 >$

   c) Give the order of 4 in $U_9$: ord(4)=

   d) Is $U_9$ a cyclic group? Explain.
Let \( \mathbb{Z}_{10, +} \) be the additive group of residue classes \( \mod 10 \).

(a) Is \( \mathbb{Z}_{10} \) a cyclic group? If so, give a generator for the group.

(b) What is the order of 4?

c) What is the inverse of 4 with respect to the group operation.

d) How many subgroups does \( \mathbb{Z}_{10} \) have?

Find a monic polynomial \( f(x) \) of degree 4 over \( \mathbb{R} \) such that \( f(2) = f'(2) = 0 \) and \( f(1 + i) = 0 \).
8) Let $G$ be the multiplicative group $G = \langle \omega \rangle$ where $\omega$ is a primitive 8-th root of unity in $\mathbb{C}$, $\omega = e^{2\pi i/8}$, ($\omega^8 = 1$).

a) Find the subgroup $\langle \omega^2 \rangle$, expressed in terms of powers of $\omega$.

b) Simplify your answer in part (a) to complex numbers in standard form $a + bi$.

c) Find another generator for $G$, other than $\omega$.

d) Find a subgroup of $G$ of order 2. (Simplify your answer. It should be very simple looking.)

9) State and prove the quadratic formula for polynomials over $\mathbb{C}$.