The point value of each problem is given in the margin. Total = 80 points.

(8)  1. a) Use the Euclidean algorithm to find a greatest common divisor of \( f(x) = x^3 + 1 \) and \( g(x) = x^2 + x \) in \( \mathbb{Q}[x] \) and then express it as a linear combination of \( f(x) \) and \( g(x) \).

\[
\begin{align*}
x^3 + 1 & \quad 1 \\
x^2 + x & \quad x^2 + x \\
- x^2 + 1 & \quad x^2 + x \\
-x^2 - x & \quad x + 1 \\
\Rightarrow \quad \text{gcd}(x^3 + 1, x^2 + x) = \text{gcd}(x^2 + x, x + 1) \\
\downarrow \\
x + 1 & \quad (x^3 + 1) - (x-1)(x^2 + x) \\
\text{gcd}(x + 1, g(x)) & \quad (x - 1)g(x) \\
\text{gcd}(x^2 + x, x + 1) & \quad = (x + 1)
\end{align*}
\]

b) Given an example of a zero divisor in \( \mathbb{Z}_6 \) and show why it is a zero divisor.

\[
[2]_6 \cdot [3]_6 = [0]_6, \quad \text{so} \quad [2]_6, [3]_6 \text{ are zero divisors. (Note, they are non-zero.)}
\]

c) How many elements are there in \( \mathbb{Z}_6 \)?

\[
6
\]

(6)  3. a) Find \( \phi(500) \), the number of integers from 1 to 500 that are relatively prime to 500.

\[
\begin{align*}
500 & = 5^3 \cdot 100 = 5^3 \cdot 10 \cdot 10 = 5^3 \cdot 2^2 \\
\phi(500) & = \phi(5^3) \phi(2^2) = (5^3 - 5^2)(2^2 - 2) = 100 - 2 = 200
\end{align*}
\]

b) How many units are there in \( \mathbb{Z}_{500} \)?

\[
200
\]
4. a) Use Fermat’s Little Theorem to find \(7^{122} \pmod{11}\).

\[ b_7 = 41, \quad 7^{10} \equiv 1 \pmod{11}. \]

Thus \(7^{122} \equiv (7^{10})^{12} \cdot 7^2 \equiv 1^{12} \cdot 49 \pmod{11} \)

\[ \equiv 49 \equiv 5 \pmod{11}. \]

b) State precisely Euler’s Theorem, the generalization of Fermat’s Little Theorem to an arbitrary modulus \(m\).

If \( \gcd(a, m) = 1 \) then \( a^{\phi(m)} \equiv 1 \pmod{m} \).

(True for any \( m > 1 \)).

5. Let \( z = 1 - i \in \mathbb{C} \).

a) Find the polar form of \(z\).

\[ |z| = \sqrt{1^2 + 1^2} = \sqrt{2}, \quad \theta = \arctan \left( \frac{-1}{1} \right) = -\frac{\pi}{4} \]

\[ z = r \, e^{i \theta} = \sqrt{2} \, e^{-i \frac{\pi}{4}} \]

(or \( z = \sqrt{2} \, e^{i \frac{5\pi}{4}} \))

b) Find the two square roots of \(z\) in \( \mathbb{C} \), expressed in polar form.

\[ z^{1/2} = \left( \sqrt{2} \, e^{-i \frac{\pi}{4}} \right)^{1/2}, \quad k = 0, 1 \]

\[ = \sqrt{2}^{1/4} \, e^{-i \left( \frac{\pi}{8} + k\pi \right)}, \quad k = 0, 1 \]

\[ = \sqrt{2}^{1/4} \, e^{-i \frac{\pi}{8}} \quad \text{or} \quad 2^{1/4} \, e^{-i \frac{3\pi}{8}} \]

Also can write \( z^{1/2} = 2^{1/4} \, e^{i \frac{\pi}{8}}, 2^{1/4} \, e^{i \frac{5\pi}{8}} \).

c) Find \(z^{14}\). Express your final answer in the form \(a + bi\).

\[ z^{14} = (\sqrt{2} \, e^{-i \frac{\pi}{4}})^{14} = 2^{7} \, e^{-i \frac{7\pi}{4}} = 2^{7} \, e^{-i \frac{\pi}{2}} = 2^{7} \, i = 128 \, i = 0 + 128 \, i \]
(20) 6. Indicate whether the following sets are closed under addition (C.A.), closed under multiplication (C.M.), Rings (Ring), Commutative rings (C.Ring), Rings with Unity (R.U.), Integral Domains (Domain), Fields (Field). Circle all correct answers on each problem.

a) \( \mathbb{N} = \text{natural numbers} \)  
C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

b) \( \mathbb{Z}_7 \)  
C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

\[
\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ac & cb \\ 0 & 0 \end{bmatrix}
\]

\[
\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ac & ad \\ 0 & 0 \end{bmatrix}
\]

c) \( \mathbb{Z}_{25} \)  
C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

d) \( M_{2,2}(\mathbb{R}) \)  
C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

e) \( \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{Z}_9 \right\} \)  
C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

f) \( \{0, [3], [6]\} \) in \( \mathbb{Z}_9 \)  
C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

g) \( \mathbb{R}^+ \cup \{0\} = \text{nonnegative reals} \)  
C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

h) \( \mathbb{Z}[x] \)  
C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

i) \( E = \{2n : n \in \mathbb{Z}\} \)  
C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field
(Here \( E = \{2n : n \in \mathbb{Z}\} \) , the set of even integers.)

j) \( \mathbb{C} \)  
C.A.  C.M.  Ring  C.Ring  R.U.  Domain  Field

(8) 7. Prove that if \( F \) is a field then \( F \) is an integral domain.

Let \( F \) be a field. Then by def, \( F \) is a commutative ring with unity. We need to prove \( F \) has no zero divisors. Let \( x, y \in F \) with \( xy = 0 \). If \( x = 0 \) we are done. If \( x \neq 0 \) then \( x \) has a mult. inverse \( x^{-1} \in F \). Then

\[
x^{-1}(xy) = x^{-1}0
\]

\[
\Rightarrow (x^{-1}x)y = 0 , \text{ by assoc. law and property that } a, 0 = 0 \text{ in any ring.}
\]

\[
\Rightarrow 1 \cdot y = 0
\]

\[
\Rightarrow y = 0
\]

Thus, either \( x = 0 \) or \( y = 0 \). QED.
8. Answer either part A or part B. (If you do both you can earn 2 extra credit points.)

A) Prove that a natural number $n$ is divisible by 9 if and only if the sum of its digits is divisible by 9.

Let $n = a_k 10^k + a_{k-1} 10^{k-1} + \cdots + a_1 10 + a_0$, written in base 10, where $a_0, a_1, \ldots, a_k$ are the base-10 digits. Then

\[ 9 \mid n \iff n \equiv 0 \pmod{9} \iff a_k 10^k + \cdots + a_0 \equiv 0 \pmod{9} \]

\[ \iff a_k + a_{k-1} + \cdots + a_0 \equiv 0 \pmod{9} \]

\[ \iff 9 \mid (a_k + a_{k-1} + \cdots + a_0) \]

B) Find digits $a, b$ such that the decimal $ab56711$ is divisible by 99. ($a, b \in \{0, 1, 2, \ldots, 9\}$.

\[ 9 \mid ab56711 \iff a + b + 5 + 6 + 7 + 1 + 1 \equiv 0 \pmod{9} \]

\[ \iff a + b = 7 \text{ or } 16, \text{ since } 0 \leq a, b \leq 9 \]

\[ 11 \mid ab56711 \iff a - b + 5 - 6 + 7 - 1 + 1 \equiv 0 \pmod{11} \]

\[ \iff a - b + 6 \equiv 0 \pmod{11} \]

\[ \iff a - b + 6 \equiv 5 \pmod{11} \]

\[ \iff a - b = 5 \text{ or } -6, \text{ since } 0 \leq a, b \leq 9. \]

Next, we test the pairs:

i) $a + b = 7$, $a - b = 5$ \Rightarrow $2a = 12$, $a = 6$, $b = 1$

ii) $a + b = 7$, $a - b = -6$ \Rightarrow $2a = 1$ \text{ can't solve}

iii) $a + b = 16$, $a - b = 5$ \Rightarrow $2a = 21$, \text{ can't solve}

iv) $a + b = 16$, $a - b = -6$ \Rightarrow $2a = 10$ \Rightarrow $a = 5$, $b = 11$

\text{not acceptable.}