The point value of each problem is given in the margin. Total = 80 points.

1. a) Use the Euclidean algorithm to find a greatest common divisor of \( f(x) = x^3 + 1 \) and \( g(x) = x^2 + x \) in \( \mathbb{Q}[x] \) and then express it as a linear combination of \( f(x) \) and \( g(x) \).

2. a) Describe the typical element in the congruence class \([2]_6 \in \mathbb{Z}_6\).

b) Give an example of a zero divisor in \( \mathbb{Z}_6 \) and show why it is a zero divisor.

c) How many elements are there in \( \mathbb{Z}_6 \)?

3. a) Find \( \phi(500) \), the number of integers from 1 to 500 that are relatively prime to 500.

b) How many units are there in \( \mathbb{Z}_{500} \)?
4. a) Use Fermat’s Little Theorem to find \( 7^{122} \pmod{11} \).

b) State precisely Euler’s Theorem, the generalization of Fermat’s Little Theorem to an arbitrary modulus \( m \).

5. Let \( z = 1 - i \in \mathbb{C} \).

a) Find the polar form of \( z \).

b) Find the two square roots of \( z \) in \( \mathbb{C} \), expressed in polar form.

c) Find \( z^{14} \). Express your final answer in the form \( a + bi \).
6. Indicate whether the following sets are closed under addition (C.A.), closed under multiplication (C.M.), Rings (Ring), Commutative rings (C.Ring), Rings with Unity (R.U.), Integral Domains (Domain), Fields (Field). Circle all correct answers on each problem.

a) $\mathbb{N}$ = natural numbers C.A. C.M. Ring C.Ring R.U. Domain Field

b) $\mathbb{Z}_7$ C.A. C.M. Ring C.Ring R.U. Domain Field

c) $\mathbb{Z}_{25}$ C.A. C.M. Ring C.Ring R.U. Domain Field

d) $M_{2,2}(\mathbb{R})$ C.A. C.M. Ring C.Ring R.U. Domain Field

e) $\left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ C.A. C.M. Ring C.Ring R.U. Domain Field

f) $\{[0],[3],[6]\}$ in $\mathbb{Z}_9$ C.A. C.M. Ring C.Ring R.U. Domain Field

g) $\mathbb{R}^+ \cup \{0\} =$ nonnegative reals C.A. C.M. Ring C.Ring R.U. Domain Field

h) $\mathbb{Z}[x]$ C.A. C.M. Ring C.Ring R.U. Domain Field

i) $\mathbb{E}$ C.A. C.M. Ring C.Ring R.U. Domain Field
(Here $\mathbb{E} = \{2n : n \in \mathbb{Z}\}$, the set of even integers.)

j) $\mathbb{C}$ C.A. C.M. Ring C.Ring R.U. Domain Field

(8) 7. Prove that if $F$ is a field then $F$ is an integral domain.
8. **Answer either part A or part B.** (If you do both you can earn 2 extra credit points.)

A) Prove that a natural number $n$ is divisible by 9 if and only if the sum of its digits is divisible by 9.

B) Find digits $a, b$ such that the decimal $ab56711$ is divisible by 99. ($a, b \in \{0, 1, 2, \ldots, 9\}$.)