INTRODUCTION TO NUMBER THEORY
Exam 2
March 27, 2009

The point value of each problem is given in the margin. Total=80 points. No calculators.

The point value of each problem is given in the margin. Standard notation is used:
\( \tau(n) = \) the number of positive divisors of \( n \).
\( \sigma(n) = \) the sum of the positive divisors on \( n \).
\( \phi(n) = \) the number of integers relatively prime to \( n \) from 1 to \( n \).
\( \mu(n) = 1 \) if \( n = 1 \), \( 0 \) if \( p^2 | n \), and \( (-1)^k \) if \( n = p_1 \ldots p_k \).

\( \text{(28)} \) 1. Short Answer:
   a) Give an example of a prime number in \( \mathbb{N} \) that factors (nontrivially) in the Gaussian integers \( \mathbb{Z}[i] \) and give the factorization.
   
   b) Give an example of a prime number in \( \mathbb{N} \) that remains prime in the Gaussian integers \( \mathbb{Z}[i] \) and explain why it can’t be factored.
   
   c) Give a prime divisor of \( 2^{44} + 1 \) bigger than 10.
   
   d) It is known that \( p = 2^{31} - 1 \) is a prime. Give a perfect number having \( p \) as a factor.
   
   e) A number less than 400 (such as 397) is tested for primality by dividing by small primes 2,3,5,7,... What is the largest prime divisor that must be tested?
   
   f) Suppose \( f \) is a multiplicative function with \( f(2) = 2 \), \( f(3) = 3 \), \( f(4) = 5 \). Find \( f(8) \) and \( f(12) \) or state that it can’t be determined.
   
   g) Evaluate \( \sum_{d|76542} \mu(d) = \) and \( \sum_{d|76542} \phi(d) = \)
(16) 2. Find the following.
   (a) The prime power factorization of 360.

   (b) $\tau(360) =$

   (c) $\sigma(360) =$

   (d) $\phi(360) =$

(9) 3. Prove one of the following.
   (a) There are infinitely many primes.
   (b) There are arbitrarily large gaps between consecutive primes.
4. A natural number $n$ is called 3-perfect if $\sigma(n) = 3n$. Find all 3-perfect numbers of the form $n = 15 \cdot 2^k$, where $k$ is a positive integer.

5. Let $f(n)$ be a multiplicative function and $F(n)$ be defined by

$$F(n) = \sum_{d|n} f(d)$$

for any natural number $n$, where the sum is over all positive divisors of $n$. Prove that $F(n)$ is multiplicative.

6. Let $F(n) = \sum_{d|n} \mu(d)\sigma(d)$.
   
   (a) Explain why $F(n)$ is a multiplicative function.

   (b) Find $F(p^k)$ for any prime power $p^k$.

   (c) Find $F(3000) =$