INTRODUCTION TO NUMBER THEORY
Exam 2
March 16, 2007

The point value of each problem is given in the margin. Total = 80 points.

The point value of each problem is given in the margin. Standard notation is used: \((a, b) = \text{GCD}, [a, b] = \text{LCM}, p^k \parallel n \text{ if } p^k | n \text{ but } p^{k+1} \nmid n.\)
\(\tau(n) = \text{the number of positive divisors of } n.\)
\(\sigma(n) = \text{the sum of the positive divisors on } n.\)
\(\phi(n) = \text{the number of integers relatively prime to } n \text{ from 1 to } n.\)
\(\mu(n) = 1 \text{ if } n = 1, 0 \text{ if } p^2 | n, \text{ and } (-1)^k \text{ if } n = p_1 \ldots p_k.\)

(9) 1. Let \(a = 2^9 \cdot 3^5 \cdot 7^2, \) \(b = 2^5 \cdot 3^5 \cdot 5^7.\) Find the following.
   (i) \((a, b) = \) The prime factorization will do!
   (ii) \([a, b] = \) Same comment.
   (iii) The value \(e\) such that \(5^e \parallel (a + b)\)

(10) 2. Use the Sieve of Eratosthenes to find all the primes between 200 and 220. (Cross out the nonprimes.) What is the largest prime divisor that must be sifted out?

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(9) 3. (a) Suppose \(a, k\) are positive integers such that \(a^k - 1\) is a prime. What can be said about \(k?\) (explain)

(b) Find two distinct prime divisors of \(2^{25} + 1.\)

(c) Give a prime divisor of \(2^{25} - 1.\)
4. Find the following.
   (a) The prime power factorization of 2500.

   (b) \( \tau(2500) = \) 

   (c) \( \sigma(2500) = \) 

   (d) \( \phi(2500) = \) 

5. Suppose that \( f \) is a multiplicative function defined on \( \mathbb{N} \) such that \( f(2) = 1, \ f(3) = 5, \ f(5) = 2, \ f(12) = 15 \). For each of the following find the value or state that it cannot be determined based on the given information.
   (a) \( f(30) = \) 

   (b) \( f(4) = \) 

   (c) \( f(50) = \)
6. Prove one of the following.
   (a) If \( n = 2^k(2^{k+1} - 1) \) and \( 2^{k+1} - 1 \) is a prime then \( n \) is perfect.
   (b) \( n \) is divisible by 9 if and only if the sum of the digits of \( n \) is divisible by 9.

7. Prove one of the following.
   (a) For any nonzero positive integers \( a, b \), \([a, b](a, b) = ab\).
   (b) If \( f \) is multiplicative and \( F(n) = \sum_{d|n} f(d) \) then \( F \) is multiplicative.
8. Let $F(n) = \sum_{d|n} \mu(d)\tau(d)$.
   
   a) Find a formula for $F(p^e)$ where $p^e$ is any prime power.

b) Find a formula for $F(n)$ if $n = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$. 