This exam is worth 160 points. The point value of each problem is given in the margin.

(12) 1. Find all integer solutions of the linear equation $17x - 52y = 1$ with $200 < x < 300$.

(6) 2. Find the order of 3 (mod 13).

(6) 3. What is the “ones” digit of the number $3^{801}$?

(12) 4. Let $a = 2^55^37^2$, $b = 2^93^45^7$. Find the following. Recall, $(a,b)=\text{GCD}$, $[a,b]=\text{LCM}$
   
   (i) $(a, b) =$ The prime factorization will do!

   (ii) Find $e$ such that $2^e|[a, b]$.

   (iii) Find $f$ such that $5^f|(b + 2a)$. 
5. Prove by induction that $4^n \equiv 1 + 3n \pmod{9}$ for any positive integer $n$.

6. Find the least positive integer $x$ satisfying the congruences

\[ x \equiv 1 \pmod{97} \text{ and } 5x \equiv 3 \pmod{9}. \] (Don’t use trial and error.)

7. a) Define what it means for a function $f$ defined on $\mathbb{N}$ to be multiplicative.

b) Suppose that $f$ is a multiplicative function such that $f(2) = 5$, $f(4) = 10$, and for any odd prime $p$, $f(p) = -1$, $f(p^2) = 2$. Calculate the following or state that it cannot be determined.

\[ f(28) = \]

\[ f(18) = \]

\[ f(54) = \]
8. In this problem you can leave your final answers as products of numbers. $\tau(n)$ is the divisor function, $\sigma(n)$ the sum of the divisors and $\phi(n)$ the Euler phi-function. Find
(a) $\tau(750) =$
(b) $\sigma(750) =$
(c) $\phi(750) =$

9. Prove the following theorem: If \( a \equiv b \pmod{m} \) and \( b \equiv c \pmod{m} \) then \( a \equiv c \pmod{m} \).

10. Evaluate the Legendre symbol \( \left( \frac{70}{11} \right) \)

11. Find a complete set of solutions (if any) of the congruence
\[ 5x^3 + 7x^2 + 15x + 14 \equiv 0 \pmod{35} \]
12. Given that the only solution of \( x^3 + x + 1 \equiv 0 \pmod{11} \) is \( x \equiv 2 \pmod{11} \), solve the congruence \( x^3 + x + 1 \equiv 0 \pmod{121} \). (Don’t use trial and error.)

13. Prove one of the following theorems:

1) There are infinitely many primes.

2) Any positive integer \( n > 1 \) can be expressed as a product of primes.
Do any three of the next six problems. Additional problems will count as extra credit at a value of 4 points per problem. (You’re best three will be worth 8 points each.)

(8) 14. The Möbius function $\mu$ is defined by $\mu(1) = 1$, $\mu(n) = (-1)^k$ if $n = p_1 p_2 \ldots p_k$, a product of $k$ distinct primes, and $\mu(n) = 0$ if $p^2|n$ for some prime $p$. Find a formula for $\sum_{d|n} \mu^2(d)$, if $n = p_1^{e_1} \ldots p_k^{e_k}$.

(8) 15. Suppose that $a, b$ are positive integers relatively prime to 10 such that the decimal expansions of $1/a$ and $1/b$ have repeating cycles of (minimal) lengths $m, n$ respectively. If $(a, b) = 1$ what is the length of the repeating cycle for $1/(ab)$? Prove your answer.

(8) 16. Prove that 5 is not a prime and that 7 is a prime in the set of Gaussian integers $\{a + bi : a, b \in \mathbb{Z}\}$
17. Find a best possible approximation of \( \frac{151}{110} \) by a fraction \( \frac{r}{s} \) with \( r \) and \( s \) less than 25.

18. Prove that if \( n = 2^k(2^{k+1} - 1) \) with \( 2^{k+1} - 1 \) a prime, then \( n \) is a perfect number.

19. Let \( p \) be an odd prime. Prove that \( -1 \) is a quadratic residue \( \pmod{p} \) if and only if \( p \equiv 1 \pmod{4} \).