The $\delta$-squared process and Fourier series of functions with multiple jumps

Emily Jennings* Charles N. Moore*† Daniel Muñiz* Ashley Toth*
Georgia Institute of Technology Kansas State University University of Florida Rollins College
Atlanta, GA 30332 Manhattan, KS 66506 Gainesville, FL 32611 Winter Park, FL 32789
ejennings3@gatech.com cnmoore@math.ksu.edu dmuniz4444@gmail.com atoth@rollins.edu

Abstract
We investigate the effects of the $\delta^2$ transform on the partial sums of Fourier series for functions with a finite number of jumps, which in general, converge slowly. Although the $\delta^2$ process is known to accelerate convergence for many sequences, we prove that in this case, the transformed series will usually fail to converge to the original function.

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1 Introduction
For a function $f$ integrable on $[-\pi, \pi]$, we define the Fourier coefficients by
\[ \hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-i nx} \, dx \]
for each integer $n$, and the $N^{th}$ partial sum of the Fourier series by
\[ S_N f(x) := \sum_{k=-N}^{N} \hat{f}(k) e^{ikx}, \quad (1) \]
where $N$ is a positive integer and $x \in [-\pi, \pi]$.

When $f$ is square integrable, $S_N f$ converges to $f$ in $L^2$. Theorems of Dini-Lipschitz, Lebesgue, and Dirichlet-Jordan give conditions for pointwise convergence (see e.g. Zygmund [17] for these). In practice, Fourier series are useful if they converge rapidly, which is usually not the case, say, for a function with jump discontinuities. For such functions it is typical that the decay of the Fourier coefficients is $O\left(\frac{1}{n}\right)$; in particular, these do not converge absolutely. In this paper we investigate the possibility of applying a well-known sequence acceleration method to the partial sums of certain slowly converging Fourier series. This continues work in Abebe, Graber and Moore [2] and Moore [10].

Given a numerical sequence $s_n$, the $\delta^2$ process transforms it to the sequence
\[ t_n := s_n - \frac{(s_{n+1} - s_n)(s_n - s_{n-1})}{(s_{n+1} - s_n) - (s_n - s_{n-1})}, \quad (2) \]
where we set $t_n = s_n$ if the denominator of the fraction is zero.

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This transform is usually attributed to Aitken [1], although the idea had appeared earlier. This transform and generalizations were studied extensively by Shanks [11], and it is for this reason this is sometimes called the Shanks transform. The $\delta^2$ process and many related transforms are discussed in Brezinski and Redivo-Zaglia [6] or Sidi [13]. The article by Tucker [15] gives several conditions on a convergent $s_n$ so the resulting $t_n$ converges to the same limit but more quickly.

In this article we will investigate what happens when the $\delta^2$ process is applied pointwise to the sequence of partial sums $S_n f(x)$ of a function which is smooth except for a finite number of jumps. Usually such functions have Fourier series which converge slowly.

Smith and Ford [14] used numerical tests to compare different methods of convergence acceleration to the partial sums of Fourier series. Using a set of five points they tested a slowly and a rapidly converging the partial sums of Fourier series. Using a set of five points they tested a slowly and a rapidly converging

Integration by parts gives the proof.

Lemma 1. Let $f$ be a piecewise $2\pi$ periodic $C^2$ function having a finite number of jump discontinuities in $(-\pi, \pi)$ at $a_1 < a_2 < \ldots < a_m$. Suppose that for each $j$, \( \lim_{x \to a_j} f(x) = f(a_j^\pm) \) and \( \lim_{x \to a_j} f'(x) = f'(a_j^\pm) \) exist and are finite. For each $j$ set $d_j = f(a_j^-) - f(a_j^0)$ and $d_j' = f'(a_j^-) - f'(a_j^0)$. Then the $N^{th}$ partial sum of the Fourier series is

\[
S_N f(x) = \hat{f}(0) + \sum_{k=1}^{N} \left[ \frac{1}{k\pi} \sum_{j=1}^{m} [d_j \sin k(x - a_j)] + \epsilon_k \right],
\]

where

\[
\epsilon_k = \frac{1}{k^2\pi} \sum_{j=1}^{m} [d_j' \cos k(x - a_j)] - \frac{\hat{f}'(k) e^{ikx} + \hat{f}''(-k)e^{-ikx}}{k^2}.
\]

Proof. Integration by parts gives the $n^{th}$ Fourier coefficient,

\[
\hat{f}(n) = \frac{1}{-2i\pi n} \sum_{j=1}^{m} [(f(a_j^-) - f(a_j^+)) e^{i n a_j}] + \frac{1}{2n^2\pi} \sum_{j=1}^{m} [(f'(a_j^-) - f'(a_j^+)) e^{-i n a_j}] - \frac{1}{n^2} \hat{f}'(n).
\]

By equation (1),

\[
S_N f(x) = \hat{f}(0) + \sum_{k=1}^{N} \hat{f}(-k)e^{-ikx} + \sum_{k=1}^{N} \hat{f}(k)e^{ikx},
\]

which simplifies to (3).
Applying the $\delta^2$ process pointwise to the sequence $\{S_N f(x)\}$, yields the sequence

$$T_N f(x) = S_N f(x) - \frac{(S_{N+1} f(x) - S_N f(x))(S_N f(x) - S_{N-1} f(x))}{(S_{N+1} f(x) - S_N f(x)) - (S_N f(x) - S_{N-1} f(x))}$$

$$= S_N f(x) - \frac{1}{(N+1)\pi} \sum_{j=1}^{m} [d_j \sin (N + 1)(x - a_j)] + \epsilon_{N+1} \left[ \frac{1}{N\pi} \sum_{j=1}^{m} [d_j \sin (x - a_j)] + \epsilon_N \right].$$

(5)

where $\epsilon_N$ is as defined in (4). Multiplying the numerator and denominator in the fraction on the right above by $N(1)^2$, we obtain the expression.

$$\frac{\sum_{j=1}^{m} [d_j \sin (N + 1)(x - a_j)] + (N + 1)\pi \epsilon_{N+1}}{N\pi \sum_{j=1}^{m} [d_j \sin (N + 1)(x - a_j)] - (N + 1)\pi \sum_{j=1}^{m} [d_j \sin (x - a_j)] + N(1^2)(\epsilon_{N+1} - \epsilon_N)}.$$

(6)

We note that $\epsilon_N \in O(\frac{1}{N^2})$. We need estimates of both the numerator and denominator of (6). We begin with two lemmas which estimate the denominator, first for $N$ fixed, then for $x$ fixed.

**Lemma 2.** For a fixed real number $x_0$, the function $g(x) = \sum_{j=1}^{m} d_j (\sin (N + 1)(x - a_j) - \sin (x - a_j))$ satisfies $|g(x)| \leq \frac{C}{N}$ on a subinterval of $[x_0, x_0 + \frac{\pi}{N+1}]$. Here $C$ is a constant which depends only on the jumps $d_j$.

**Proof.** Note that

$$g\left(x_0 + \frac{\pi}{N+1}\right) = \sum_{j=1}^{m} d_j \left(- \sin (N + 1)(x_0 - a_j) + \sin (N(x_0 - a_j) - \frac{\pi}{N+1})\right)$$

$$= \sum_{j=1}^{m} d_j \left(- \sin (N + 1)(x_0 - a_j) + \sin (N(x_0 - a_j)) \cos \frac{\pi}{N+1} - \cos (N(x_0 - a_j)) \sin \frac{\pi}{N+1}\right)$$

$$= - \sum_{j=1}^{m} d_j (\sin (N + 1)(x_0 - a_j) - \sin (x_0 - a_j))$$

$$+ \sum_{j=1}^{m} d_j \left[ \sin (N(x_0 - a_j)) \cos \left(\frac{\pi}{N+1} - 1\right) - \cos (N(x_0 - a_j)) \sin \frac{\pi}{N+1}\right]$$

$$= - g(x_0) + e(x_0, N),$$

where $e(x_0, N) = \sum_{j=1}^{m} d_j \left[ \sin (N(x_0 - a_j)) \cos \left(\frac{\pi}{N+1} - 1\right) - \cos (N(x_0 - a_j)) \sin \frac{\pi}{N+1}\right]$. We estimate: $|e(x_0, N)| \leq \sum_{j=1}^{m} |d_j| \left(\frac{2\pi}{N+1}\right) \leq \frac{C}{4N}$.

Case 1: If $|g(x_0)| \leq \frac{C}{2N}$, then since $|g'| \leq (\sum_{j=1}^{m} |d_j|)(2N + 1)$, $|g| \leq \frac{C}{N}$ on a subinterval of $[x_0, x_0 + \frac{\pi}{N+1}]$ of length approximately $\frac{1}{N^2}$.

Case 2: If $|g(x_0)| > \frac{C}{2N}$, then $g(x_0)$ and $g\left(x_0 + \frac{\pi}{N+1}\right)$ have opposite signs: Say $g(x_0) > 0$. Then $g(x_0) > \frac{C}{2N}$ and $-g(x_0) < -\frac{C}{2N}$, so

$$g\left(x_0 + \frac{\pi}{N+1}\right) = -g(x_0) + e(x_0, N) \leq -\frac{C}{2N} + \frac{C}{4N} = -\frac{C}{4N},$$

and thus, $g(x) = 0$ somewhere on the interval. As in Case 1, this implies that $g$ is bounded above by $\frac{C}{N}$ for some constant $C$ on an interval of length approximately $\frac{1}{N^2}$. 

$\Box$
Now set \( \alpha_N = \sum_{j=1}^{m} d_j \cos N a_j, \beta_N = \sum_{j=1}^{m} d_j \sin N a_j, A_N = \sqrt{\alpha_N^2 + \beta_N^2} \) and \( \phi_N = \arctan \left( \frac{\beta_N}{\alpha_N} \right) \). (If \( \alpha_N = 0, \beta_N \geq 0 \), set \( \phi_N = \frac{\pi}{2} \), and if \( \alpha_N = 0, \beta_N < 0 \), set \( \phi_N = -\frac{\pi}{2} \).) Then for \( x \in [-\pi, \pi] \),

\[
\sum_{j=1}^{m} d_j \sin N(x - a_j) = \sum_{j=1}^{m} d_j (\sin Nx \cos Na_j - \cos Nx \sin Na_j)
= \alpha_N \sin Nx - \beta_N \cos Nx = A_N \sin(Nx - \phi_N).
\]

In Lemma 4 below, we will assume that for each \( j \), \( a_j = \frac{p_j}{q_j} \pi \) with \( \frac{p_j}{q_j} \) in lowest terms. Then for \( L = 2 \text{lcm}\{q_j\} \), the sequences \( \alpha_n, \beta_n, A_n \) and \( \phi_n \) have period \( L \), that is, \( \alpha_{n+L} = \alpha_n \) for all \( n \), etc. We first need a result from elementary number theory.

**Lemma 3.** (Chebyshev; see [9], pg. 39) For an arbitrary irrational number \( a \) and arbitrary real number \( m \), there exists an infinite sequence of positive integers \( k \) and \( m \) such that \( |ka - m - \mu| < \frac{3}{k} \).

**Lemma 4.** Suppose that each \( a_n \) is a rational multiple of \( \pi \), that is, \( a_j = \frac{p_j}{q_j} \pi \) which we assume to be in lowest terms. Suppose \( x = 2a \pi \), where \( a \in [-.5, .5] \) is an irrational number. Fix \( J \in \{1, ..., L\} \) which has \( A_J A_{J+1} \neq 0 \). Then there are an infinite number of integers \( n = kL + J \) such that \( |A_{n+1} \sin ((n+1)x - \phi_{n+1}) - A_n \sin (nx - \phi_n)| \leq \max\{A_J, A_{J+1}\} \frac{24L^2\pi}{n} \).

**Proof.** Note that for \( n = kL + J \), we have \( A_n = A_J, A_{n+1} = A_{J+1}, \phi_n = \phi_J, \) and \( \phi_{n+1} = \phi_{J+1} \). Thus, we must show that

\[
|A_{J+1} \sin ((kL + J + 1)x - \phi_{J+1}) - A_J \sin ((kL + J)x - \phi_J)| \leq \frac{24L^2\pi \max\{A_J, A_{J+1}\}}{kL + J}
\]

for an infinite number of \( k \). Set \( h(\theta) = A_{J+1} \sin (\theta + x - \phi_{J+1}) - A_J \sin (\theta - \phi_J) \). Since \( \int_{-\pi}^{\pi} h(\theta) \, d\theta = 0 \) and \( h \) is continuous, there exists \( \theta^* \in [-\pi, \pi] \) such that \( h(\theta^*) = 0 \).

By Lemma 3 with \( \mu = \frac{\theta^* - Jx}{2L\pi} \), there are an infinite number of positive integers \( k \) and \( m \) such that

\[
|ka - m - \frac{\theta^* - Jx}{2L\pi}| < \frac{3}{k}.
\]

Multiplying by \( 2L\pi \) and substituting \( x = 2\pi a \), we obtain

\[
|(kL + J)x - \theta^* - 2Lm\pi| < \frac{6L\pi}{k} \leq \left( \frac{6L\pi}{kL + J} \right) \left( \frac{L(k + 1)}{k} \right) \leq \frac{12L^2\pi}{kL + J}.
\]

Hence, since \( h \) is \( 2\pi \) periodic,

\[
|h((kL + J)x)| = |h((kL + J)x) - h(\theta^*)| \leq ||h'|| \infty \left( \frac{12L^2\pi}{kL + J} \right) \leq \frac{24L^2\pi \max\{A_J, A_{J+1}\}}{kL + J},
\]

which gives (7).

**Lemma 5.** Suppose \( f \) is as in Lemma 1 and suppose that \( A_N A_{N+1} \neq 0 \). Let \( \eta > 0 \). Then if \( N \) is sufficiently large, except for a set \( E_{N, \eta} \), at any \( x \) where \( |g(x)| \leq \frac{C}{N} \), the numerator of (6) is bounded below by \( c \min\{A_N, A_{N+1}\}^2 - \frac{C}{N} \). Here \( c \) and \( C \) do not depend on \( x \) or \( N \). \( E_{N, \eta} \) is comprised of two or three intervals and has \( |E_{N, \eta}| < \eta \).

**Proof.** With the notation as above, we estimate the numerator of (6):

\[
A_N \sin (Nx - \phi_N) A_{N+1} \sin ((N+1)x - \phi_{N+1}) + (N+1)\epsilon_{N+1} A_N \sin (Nx - \phi_N) + N\epsilon_N A_{N+1} \sin ((N+1)x - \phi_{N+1}) + N(N+1)\pi^2 \epsilon_{N+1} \leq A_N \sin (Nx - \phi_N) A_{N+1} \sin ((N+1)x - \phi_{N+1}) - \frac{c_o}{N+1} - \frac{c_o}{N} + \frac{e_o^2}{N(N+1)}
\]

(8)
for some \( c_0 \in \mathbb{R} \).

Suppose \( x \) is a point at which \(|A_N \sin (Nx - \phi_N) - A_{N+1} \sin ((N+1)x - \phi_{N+1})| \leq \frac{C}{N} \). Squaring both sides and rearranging yields
\[
A_N A_{N+1} \sin ((N+1)x - \phi_{N+1}) \sin (Nx - \phi_N) \\
\geq \frac{1}{2} \min\{A_N, A_{N+1}\}^2 \left[ \sin^2 ((N+1)x - \phi_{N+1}) + \sin^2 (Nx - \phi_N) \right] - \left( \frac{C}{N} \right)^2. \tag{9}
\]

Set \( y = x - \frac{\phi_{N+1} + \phi_N}{2N+1} \) and \( \phi = \frac{N\phi_{N+1} - (N+1)\phi_N}{2N+1} \). Then
\[
\sin^2 ((N+1)x - \phi_{N+1}) + \sin^2 (Nx - \phi_N) = \sin^2 ((N+1)y - \phi) + \sin^2 (Ny + \phi) \\
= \left[ \sin \left( Ny + \frac{y}{2} \right) \cos \left( \frac{y}{2} - \phi \right) + \cos \left( Ny + \frac{y}{2} \right) \sin \left( \frac{y}{2} - \phi \right) \right]^2 + \\
\left[ \sin \left( Ny + \frac{y}{2} \right) \cos \left( \frac{y}{2} - \phi \right) - \cos \left( Ny + \frac{y}{2} \right) \sin \left( \frac{y}{2} - \phi \right) \right]^2 \\
= \sin^2 \left( Ny + \frac{y}{2} \right) \cos^2 \left( \frac{y}{2} - \phi \right) + \cos^2 \left( Ny + \frac{y}{2} \right) \sin^2 \left( \frac{y}{2} - \phi \right).
\]

Let \( \eta > 0 \). Consider the set
\[
F_{N, \eta} = [-\pi, \pi] - \{ (2\phi - \eta, 2\phi + \eta) \cup (2\phi - \pi - \eta, 2\phi - \pi + \eta) \cup (2\phi + \pi - \eta, 2\phi + \pi + \eta) \}.
\]

Then \( \gamma = \sin^2 \left( \pm \frac{\eta}{2} \right) = \cos^2 \left( \frac{\eta}{2} \pm \eta \right) \) is a lower bound for \( \sin^2 \left( \frac{y}{2} - \phi \right) \) and \( \cos^2 \left( \frac{y}{2} - \phi \right) \), on \( F_{N, \eta} \), and hence
\[
\sin^2 \left( Ny + \frac{y}{2} \right) \cos^2 \left( \frac{y}{2} - \phi \right) + \cos^2 \left( Ny + \frac{y}{2} \right) \sin^2 \left( \frac{y}{2} - \phi \right) \geq \gamma \quad \text{on} \quad F_{N, \eta}.
\]

Put \( \varphi = \phi_{N+1} - \phi_N \). For large enough \( N \), independent of \( x, y, \phi_N \) and \( \phi_{N+1} \), \(|x - y| \leq \frac{\eta}{4}, |2\phi - \varphi| \leq \frac{\eta}{4} \), and so for \( x \) in the set
\[
E_{N, \eta} = [-\pi, \pi] - \{ (\varphi - \frac{\eta}{2}, \varphi + \frac{\eta}{2}) \cup (\varphi - \frac{\eta}{2}, \varphi - \pi + \frac{\eta}{2}) \cup (\varphi + \pi - \frac{\eta}{2}, \varphi + \pi + \frac{\eta}{2}) \}
\]

we have \( \sin^2 ((N+1)x - \phi_{N+1}) + \sin^2 (Ny - \phi_N) \geq \gamma \). Combining this, (8) and (9) gives, for large enough \( N \), a lower bound of \( \frac{\pi}{2} \min\{A_N, A_{N+1}\}^2 - \frac{C}{N} \) for (8) at any \( x \in E_{N, \eta} \) at which \(|g(x)| \leq C/N \). \( \square \)

**Theorem 1.** Let \( f \) be a piecewise \( C^2 \) function on \([-\pi, \pi]\) having a finite number of jump discontinuities at \( a_1 < a_2 < \ldots < a_m \in (-\pi, \pi) \), and suppose \( f(a_j^+) \) and \( f'(a_j^-) \) exist and are finite.

(a) Suppose \( N \) is a sufficiently large positive integer with \( A_N A_{N+1} \neq 0 \). Then there exists intervals on which \(|T_N f(x) - S_N f(x)| \geq m \min\{A_N, A_{N+1}\}^2 - \frac{C}{N} \), where \( m \) and \( C \) are constants which do not depend on \( N \) or \( x \).

(b) Suppose that the \( a_j \) are rational multiples of \( \pi \), \( a_j = \frac{b_j}{q_j} \pi \) (in lowest terms), so that the sequence \( A_n \) has period \( L = 2 \text{lcm}\{q_j\} \). Suppose that there exists a \( J \in \{1, \ldots, L\} \) such that \( A_J A_{J+1} \neq 0 \). Then \( T_N f \) does not converge to \( S_N f \) uniformly.

**Proof.** Using Lemma 2, we find for each interval \([-\pi + \frac{\pi j}{N+1}, -\pi + \frac{\pi (j+1)}{N+1}] \), \( j = 0, 1, \ldots, 2N+1 \), a subinterval on which \(|g(x)| \leq \frac{C}{N} \), so that for \( x \) in these intervals the denominator of (6) can be estimated by \( N\pi |g(x)| + \pi \sum_{j=1}^m |d_j| + C\pi^2 \). By Lemma 5, on these intervals intersected with \( E_{N, \eta} \), the numerator is bounded below by \( c \min\{A_N, A_{N+1}\}^2 - \frac{C}{N} \) if \( N \) is large enough. Thus, on these intervals, we have a lower bound for the size of (6) and (a) follows.

For (b), note that \( A_{kL+j} = A_J, A_{kL+j+1} = A_{J+1} \). Then for \( k = 1, 2, \ldots, (a) \) gives a constant \( m \) and points \( x \) (depending on \( k \)) at which \(|T_k f(x) - S_k f(x)| \geq m \min\{A_J, A_{J+1}\}^2 - \frac{C}{kL+1} \), and hence when \( k \) is large enough shows (b). \( \square \)

**Theorem 2.** Suppose \( x = 2a\pi \), where \( a \) is an irrational number, and suppose that there exists \( J \in \{1, \ldots, L\} \) such that \( A_J A_{J+1} \neq 0 \). Then \( \{T_N f(x)\} \) fails to converge.
Proof. For $x = 2\pi a$, $a$ irrational, Lemma 4 immediately gives an infinite number of $k$ (which depend on $x$) for which the denominator of (6) with $N = kL + J$ is bounded. Then Lemma 5 gives a lower bound for the numerator at these points.

Remarks. Thus, we have shown that under the hypotheses of the Theorems, a transformed partial sum will always have “spikes”, and that at each $x = 2\pi a$ a irrational, these spikes will occur at $x$ for an infinite number of $N$ as $N \to \infty$.

Notice that if just one of $A_N$ and $A_{N+1}$ are 0 then in (6) the numerator is at most $O(N^{-1})$, whereas the denominator seems to be either $O(N)$, or bounded, or maybe smaller with sufficient cancellation. If both $A_N$ and $A_{N+1}$ are 0, then the numerator of (6) seems to be $O(N^{-2})$ and the denominator seems to be bounded, or maybe smaller with sufficient cancellation. This is very imprecise, but in either case it seems likely that the numerator of (6) is much smaller than the denominator and thus, in these cases, we would expect that graphs of the $S_N f$ and $T_N f$ to look nearly identical for large $N$.

The condition $A_N \neq 0$ is equivalent to $\sum_{j=1}^m d_j e^{ina_j} \neq 0$. For a fixed $m$, fix $\theta \in [-\pi, \pi]$, place all the $a_j$ at the $m$th roots of $e^{i\theta}$, and set all $d_j$ equal so that $\sum_{j=1}^m d_j e^{ina_j} = 0$ for all $n$ except multiples of $m$. Thus, for a function with equal jumps at these $a_j$, the Theorems will not apply. Other constructions having $A_N A_{N+1} = 0$ for every $N$ are also possible.

Consider the function $f_1$ which is given by $f(x) = 1$, if $-1 \leq x \leq 1$ and 0 otherwise. Figure 1 shows the 20th and 30th terms of the transformed partial sums, that is, $T_{20} f_1$ and $T_{30} f_1$. Define $f_2$ by

$$f_2(x) = \begin{cases} \frac{3}{\pi} x + 1 & \text{if } x \in [-\pi, -\frac{2\pi}{3}] \\ \frac{2}{\pi} x + 1 & \text{if } x \in (-\frac{2\pi}{3}, 0) \\ 0 & \text{if } x \in (0, \frac{2\pi}{3}) \\ \frac{3}{\pi} x - 1 & \text{if } x \in (\frac{2\pi}{3}, \pi] \end{cases}$$

which is a function with jumps of 1 at points $\gamma$, where $e^{i\gamma}$ is a cube root of 1. The graphs of $S_{20} f_2$ and $T_{20} f_2$ are shown in figure 2. Here, as discussed above, our theorems do not apply, and indeed it is difficult to distinguish the graphs. Finally, define $f_3$ by

$$f_3(x) = \begin{cases} \frac{3}{\pi} x + 1 & \text{if } x \in [-\pi, -\frac{2\pi}{3}] \\ \frac{3}{\pi} x + \frac{1}{2} & \text{if } x \in (-\frac{2\pi}{3}, 0) \\ 0 & \text{if } x \in (0, \frac{2\pi}{3}) \\ \frac{3}{\pi} x - 1 & \text{if } x \in (\frac{2\pi}{3}, \pi] \end{cases}$$

This has jumps at the same points as $f_2$, but the jumps are not equal, and it is easy to check that Theorem 2 does apply. Figure 3 gives the graphs of $S_{20} f_3$ and $T_{20} f_3$.

![Figure 1: $T_{20} f_1$ and $T_{30} f_1$.](image-url)
3 Conclusions

In most cases, the $\delta^2$ process destroys the convergence of the partial sums of the Fourier series with jumps at rational multiples of $\pi$. Only if $A_N A_{N+1} = 0$ for every $N$ does the $\delta^2$ process not destroy convergence, and we have no evidence it actually improves convergence in this case. In the case of analytic functions, applying the $\delta$ squared process results in a Padé approximation (see e.g. [4]). Padé approximations can also be produced using continued fractions, which is one approach to approximation theorems such as that of Chebyshev, and many authors have illuminated these connections. Here we have made use of the Chebyshev theorem on rational approximation in a different way, and it would be interesting to see if these ideas are connected, and to have a deeper understanding of this circle of ideas in our the context. Although, as noted above, there has been some success accelerating Fourier series of some functions using the complex $\varepsilon$ algorithm, that theory is far from complete, and we hope that continued investigation of these methods will continue to develop a more complete theory.
References


