Facilitate Meaningful Mathematical Discourse

*Thursday, June 19*

Adapted from Peg Smith “Productive Discussions of Cognitively Challenging Mathematics tasks”

M4 Conference – Omaha, NE

April 21, 2014
WARM UP PROBLEM

A candy jar contains 5 Jolly Ranchers (squares) and 13 Jawbreakers (circles). Suppose you had a new candy jar with the same ratio of Jolly Ranchers to Jawbreakers, but it contained 100 Jolly Ranchers. How many Jawbreakers would you have? Explain how you know.

• Please try to do this problem in as many ways as you can, both correct and incorrect
• If done, share your work with a neighbor.
COMPARING TWO MATHEMATICAL TASKS

Task A
A candy jar contains 5 Jolly Ranchers (squares) and 13 Jawbreakers (circles). Suppose you had a new candy jar with the same ratio of Jolly Ranchers to Jawbreakers, but it contained 100 Jolly Ranchers.

How many Jawbreakers would you have? Explain how you know.

Task B
Find the value of the unknown in each of the proportions shown below.

1. \( \frac{5}{2} = \frac{y}{10} \)
2. \( \frac{n}{8} = \frac{3}{12} \)
3. \( \frac{5}{20} = \frac{3}{d} \)
4. \( \frac{a}{24} = \frac{7}{8} \)
5. \( \frac{30}{6} = \frac{b}{7} \)
6. \( \frac{3}{x} = \frac{4}{28} \)
Implement tasks that promote reasoning and problem solving

• Mathematics Teaching Practice from *Principles to Action* page 17-24

• Focus on p. 23-24 “Teacher and Student Actions” and discuss how the actions apply to these two tasks.
<table>
<thead>
<tr>
<th>Lower-Level Demands</th>
<th>Higher-Level Demands</th>
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</thead>
<tbody>
<tr>
<td><strong>Memorization</strong></td>
<td></td>
</tr>
<tr>
<td>• involve either reproducing previously learned facts, rules, formulae or definitions OR committing facts, rules, formulae or definitions to memory.</td>
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</tr>
<tr>
<td>• cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure.</td>
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<tr>
<td>• are not ambiguous. Such tasks involve exact reproduction of previously-seen material and what is to be reproduced is clearly and directly stated.</td>
<td></td>
</tr>
<tr>
<td>• have no connection to the concepts or meaning that underlie the facts, rules, formulae or definitions being learned or reproduced.</td>
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<tr>
<td><strong>Procedures Without Connections</strong></td>
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<tr>
<td>• are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instruction, experience, or placement of the task.</td>
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<tr>
<td>• require limited cognitive demand for successful completion. There is little ambiguity about what needs to be done and how to do it.</td>
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<tr>
<td>• have no connection to the concepts or meaning that underlie the procedure being used.</td>
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<tr>
<td>• are focused on producing correct answers rather than developing mathematical understanding.</td>
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<tr>
<td>• require no explanations or explanations that focuses solely on describing the procedure that was used.</td>
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<tr>
<td><strong>Procedures With Connections</strong></td>
<td></td>
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<tr>
<td>• focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas.</td>
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<tr>
<td>• suggest pathways to follow (explicitly or implicitly) that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts.</td>
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<tr>
<td>• usually are represented in multiple ways (e.g., visual diagrams, manipulatives, symbols, problem situations). Making connections among multiple representations helps to develop meaning.</td>
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<tr>
<td>• require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with the conceptual ideas that underlie the procedures in order to successfully complete the task and develop understanding.</td>
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<tr>
<td><strong>Doing Mathematics</strong></td>
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<tr>
<td>• require complex and non-algorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example).</td>
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<tr>
<td>• require students to explore and understand the nature of mathematical concepts, processes, or relationships.</td>
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<tr>
<td>• demand self-monitoring or self-regulation of one's own cognitive processes.</td>
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<tr>
<td>• require students to access relevant knowledge and experiences and make appropriate use of them in working through the task.</td>
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<tr>
<td>• require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions.</td>
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<tr>
<td>• require considerable cognitive effort and may involve some level of anxiety for the student due to the unpredictable nature of the solution process required.</td>
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HOW EXPERT DISCUSSION FACILITATION IS CHARACTERIZED

Skillful improvisation

• Diagnose students’ thinking on the fly
• Fashion responses that guide students to evaluate each others’ thinking, and promote building of mathematical content over time

Requires deep knowledge of:

• Relevant mathematical content
• Student thinking about it and how to diagnose it
• Subtle pedagogical moves
• How to rapidly apply all of this in specific circumstances
Discussion Practices

0. Setting Goals and Selecting Tasks
1. Anticipating (e.g., Fernandez & Yoshida, 2004; Schoenfeld, 1998)
2. Monitoring (e.g., Hodge & Cobb, 2003; Nelson, 2001; Shifter, 2001)
3. Selecting (e.g., Lampert, 2001; Stigler & Hiebert, 1999)
4. Sequencing (e.g., Schoenfeld, 1998)
5. Connecting (e.g., Ball, 2001; Brendehur & Frykholm, 2000)
Setting Goals

It involves:
• Identifying what students are to know and understand about mathematics
• Being as specific as possible so as to establish a clear target for instruction

It is supported by:
• Thinking about what students will come to know and understand rather than only on what they will do
• Consulting resources that can help in unpacking big ideas in mathematics
• Working in collaboration with other teachers
**Implied Goal**
Students will be able to solve the task correctly using one of a number of viable strategies and realize that there are several different and correct ways to solve the task.

**Possible Goals**
- Quantities that are in a proportional (multiplicative) relationship grow at a constant rate.
- The unit rate specifies the relationship between the two quantities; it answers the question how many times larger one quantity is in comparison to the other.
- The constant of proportionality (unit rate) can be identified in a table, graph, equation, and verbal descriptions of proportional relationships.
Establish Mathematics Goals to focus learning

- Mathematics Teaching Practice from *Principles to Action* page 12-16
- Focus on p. 16 “Teacher and Student Actions” and discuss how the actions apply to this stage of the monitoring strategy
Selecting a Task

It involves:

• Identifying a mathematical task that is aligned with the lesson goals

• Making sure the task is rich enough to support a discussion (i.e., a cognitively challenging/high level task)

It is supported by:

• Setting a clear and explicit goal for learning

• Using the Task Analysis Guide which provides a list of characteristics of tasks at different levels of cognitive demand

• Working in collaboration with colleagues
1. Anticipating

It involves considering:

• The array of strategies that students might use to approach or solve a challenging mathematical task
• How to respond to what students produce
• Which strategies will be most useful in addressing the mathematics to be learned

It is supported by:

• Doing the problem in as many ways as possible
• Discussing the problem with other teachers
• Drawing on relevant research
• Documenting student responses year to year
A candy jar contains 5 Jolly Ranchers (squares) and 13 Jawbreakers (circles). Suppose you had a new candy jar with the same ratio of Jolly Ranchers to Jawbreakers, but it contained 100 Jolly Ranchers. How many Jawbreakers would you have? Explain how you know.

• Please try to do this problem in as many ways as you can, both correct and incorrect
• If done, share your work with a neighbor.
Jawbreakers & jolly ranchers: Anticipated solutions

- **Unit Rate**—Find the number of Jawbreakers for each Jolly Rancher (1 JR: 2.6 JB) then multiply both by 100.

- **Scale Factor**—Find that the number of JB (100) is 20 times the original amount (5) so the number of JR must be 20 times the original amount (13).

- **Scaling Up**—Increasing the number of JR and JB by continuing to add 5 to the JR and 13 to the JB until you reach the desired number of JR (100).

- **Incorrect Additive**—Find that the number of JB has increased by 95 (5 + 95 = 100) so the number of JR must also increase by 95 (13 + 95 = 108)
2. monitoring

It involves:

- Circulating while students work on the problem and watching and listening
- Recording interpretations, strategies, and points of confusion
- Asking questions to get students back “on track” or to advance their understanding

It is supported by:

- Anticipating student responses beforehand
- Carefully listening and asking probing questions
- Using recording tools
Support productive struggle in learning Mathematics

• Mathematics Teaching Practice from *Principles to Action* page 48-52

• Focus on p. 52 “Teacher and Student Actions” and discuss how the actions apply to this phase of promoting productive discussion – monitoring tool strategy
<table>
<thead>
<tr>
<th>Strategy</th>
<th>Who and What</th>
<th>Order</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>List the different solution paths you anticipated</td>
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## Monitoring tool

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<td><strong>Scale Factor</strong></td>
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<td><strong>Scaling Up</strong></td>
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<td><strong>Incorrect Additive</strong></td>
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<tr>
<td><strong>OTHER –</strong></td>
<td>Ellen, Adam (drew candy jars-focused on addition)</td>
<td></td>
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<tr>
<td></td>
<td>Alicia, Max (used green and red cubes-moved from adding to multiplying)</td>
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3. Selecting

It involves:

• Choosing particular students to present because of the mathematics available in their response

• Making sure that over time all students are seen as authors of mathematical ideas and have the opportunity to demonstrate competence

• Gaining some control over the content of the discussion (no more “who wants to present next?”)

It is supported by:

• Anticipating and monitoring

• Planning in advance which types of responses to select
Lesson Goals:

• Quantities that are in a proportional (multiplicative) relationship grow at a constant rate.

• The unit rate specifies the relationship between the two quantities; it answers the question how many times larger one quantity is in comparison to the other.

• The constant of proportionality (unit rate) can be identified in a table, graph, equation, and verbal descriptions of proportional relationships.

What students do you think should be selected and to accomplish the goals?
## Monitoring tool

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<td><strong>OTHER</strong>—Ratio Table; Ricardo and Melissa—scaled up but did not increment by the same amount; Use multiplication and addition</td>
<td>Ellen, Adam (drew candy jars- focused on addition) Alicia, Max (used green and red cubes--moved from adding to multiplying)</td>
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4. Sequencing

It involves:

• Purposefully ordering presentations so as to make the mathematics accessible to all students

• Building a mathematically coherent story line

It is supported by:

• Anticipating, monitoring, and selecting

• During anticipation work, considering how possible student responses are mathematically related
Monitoring tool – in what order would you sequence the responses?

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**Possible sequencing**

1. Jordan – incorrect additive approach *(raises the issue about how the quantities are related)*
2. Kamiko – used a table to scale up *(although she used repeated addition, it can be related to multiplication and to the other strategies)*
3. Owen – scale factor *(clearly shows that the ratio is preserved and that the new jar is 20 times the original)*
4. Jerry – unit rate *(shows the multiplicative relationship between the two quantities)*
Elicit and Use evidence of student thinking

• Mathematics Teaching Practice from *Principles to Action* page 53-56. Read these pages.
• Focus on p. 56 “Teacher and Student Actions” and discuss how the actions apply to this stage of the monitoring strategy
5. connecting

It involves:

• Encouraging students to make mathematical connections between different student responses
• Making the key mathematical ideas that are the focus of the lesson salient

It is supported by:

• Anticipating, monitoring, selecting, and sequencing
• During planning, considering how students might be prompted to recognize mathematical relationships between responses
Possible connections

1. Jordan
   100 JR is 95 more than the 5 we started with. So we will need 95 more JB than the 13 we started with.
   
   \[5 \text{ JR} + 95 \text{ JR} = 100 \text{ JR}\]
   \[13 \text{ JB} + 95 \text{ JB} = 108 \text{ JB}\]

2. Kamiko
   We just kept adding 5 to the JR column and 13 to the JB column. We stopped when we got to 100 JR. So it has to be 260 JB.

3. Owen
   You have 20 more XS than the 20 for the first 20 months.
   
   So 20 XS + 20 XS = 40 XS

4. Jerry
   Since the ratio is 5 JR for 13 JB, for each JR you would have 2 JB; that would use up 10 JB. So you have three JB left over. So we had to distribute the three JB to the 5 JR. \[3 \div 5 = .6\] so that would give the ratio of 1 JR to 2.6 JB. So then you just multiply 1 and 2.6 each by 100.

Jordan and Kamiko both used addition but got different answers. How is that possible? Who got the correct answer? Why?
Possible connections

1. Jordan

100 JR is 95 more than the 5 we started with. So we will need 95 more JB than the 13 we started with.

\[5 \text{ JR} + 95 \text{ JR} = 100 \text{ JR}\]

13 JB

We just went down the JR column and 13 to the JB column. We stopped when we got to 100 JR. So it has to be 260 JB.

2. Jerry

Since the ratio is 5 JR for 13 JB, for each JR you would have 2 JB; that would use up 10 JB. So you have three JB left over. So we had to distribute the three JB to the 5 JR. \[3 \div 5 = .6\] so that would give the ratio of 1 JR to 2.6 JB. So then you just multiply 1 and 2.6 each by 100.

3. Owen

You have to multiply the five JR by 20 to get 100, so you’d also have to multiply the 13 JB by 20 to get 260. So it has to be 260.

How are Jerry and Owen’s strategies the same and how are they different?
Possible connections

3. Owen
You have to multiply the five JR by 20 to get 100, so you’d also have to multiply the 13 JB by 20 to get 260. So it has to be 260.

4. Jerry
Since the ratio is 5 JR for 13 JB, for each JR you would have 2 JB; that would use up 10 JB. So you have three JB left over. So we had to distribute the three JB to the 5 JR. $3 \div 5 = 0.6$ so that would give the ratio of 1 JR to 2.6 JB. So then you just multiply 1 and 2.6 each by 100.

2. Kamiko
We just kept adding 5 to the JR column and 13 to the JB column. We stopped when we got to 100 JR. So it has to be 260 JB.

Where is the unit rate in Kamiko’s table?

$5\text{ JR} + 95 \text{ JR} = 100 \text{ JR}$

$13 \text{ JB} + 95 \text{ JB} = 108 \text{ JB}$
Possible connections

1. Jordan

100 JR is 95 more than the 5 we started with. So we will need 95 more JB than the 13 we started with.

5 JR + 95 JR = 100 JR
13 JB + 95 JB = 108 JB

We just added 90 JR to the 5 JR column and 15 to the 13 JB column. We stopped when we got to 100 JR. So it has to be 260 JB.

2. How are Jerry and Owen’s strategies the same and how are they different?

3. Owen

You have to multiply the five JR by 20 to get 100, so you’d also have to multiply the 13 JB by 20 to get 260. So it has to be 260.

4. Jerry

Since the ratio is 5 JR for 13 JB, for each JR you would have 2 JB; that would use up 10 JB. So you have three JB left over. So we had to distribute the three JB to the 5 JR. $3 \div 5 = .6$ so that would give the ratio of 1 JR to 2.6 JB. So then you just multiply 1 and 2.6 each by 100.
Use and connect mathematical representations

• Mathematics Teaching Practice from *Principles to Action* page 24-29.
• Focus on p. 29 “Teacher and Student Actions” and discuss how the actions apply to this stage of the monitoring strategy
Mathematics teaching practices addressed:

- Implement tasks that promote reasoning and problem solving
- Establish Mathematics Goals to focus learning
- Support productive struggle in learning Mathematics
- Elicit and Use evidence of student thinking
- Use and connect mathematical representations
- Build procedural fluency from conceptual understanding
- Pose Purposeful Questions (we will focus on this in a later day of the Academy)

The really big overarching teaching practice today:
Facilitate meaningful discourse
Facilitate meaningful mathematical discourse

• Read pages 29-35 and reflect
• Consider a task you could use with this monitoring tool