THE GOLDEN RATIO AND THE FIBONACCI SEQUENCE

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Everything is Golden

- Golden Ratio
- Golden Proportion
- Golden Relation
- Golden Rectangle
- Golden Spiral
- Golden Angle
Geometric Growth, (Exponential Growth):

\[ r = \text{growth rate or common ratio} \]

- **Example.** Start with 6 and double at each step: \( r = 2 \)

  \[ 6, 12, 24, 48, 96, 192, 384, \ldots \]

  Differences:

- **Example** Start with 2 and triple at each step: \( r = 3 \)

  \[ 2, 6, 18, 54, 162, \ldots \]

  Differences:

**Rule:** The differences between consecutive terms of a geometric sequence grow at the same rate as the original sequence.
Differences and ratios of consecutive Fibonacci numbers:

1 1 2 3 5 8 13 21 34 55 89

Is the Fibonacci sequence a geometric sequence?

Let's examine the ratios for the Fibonacci sequence:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>8</th>
<th>13</th>
<th>21</th>
<th>34</th>
<th>55</th>
<th>89</th>
</tr>
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<td>34</td>
<td>55</td>
<td>89</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1.500</td>
<td>1.667</td>
<td>1.600</td>
<td>1.625</td>
<td>1.615</td>
<td>1.619</td>
<td>1.618</td>
<td>1.618</td>
<td>1.618</td>
</tr>
</tbody>
</table>

What value is the ratio approaching?
The Golden Ratio, \( \phi = 1.61803398\ldots \)

- The Golden Ratio is (roughly speaking) the growth rate of the Fibonacci sequence as \( n \) gets large.

Euclid (325-265 B.C.) in *Elements* gives first recorded definition of \( \phi \).

Next try calculating \( \frac{\phi^n}{F_n} \).

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
n & 1 & 2 & 3 & 4 & 5 & 6 & 10 & 12 \\
\hline
\phi^n & 1.618 & 2.618 & 2.118 & 2.285 & 2.218 & 2.243 & 2.236 & 2.236 \\
F_n & & & & & & & & \\
\hline
\end{array}
\]

\[
\frac{\phi^n}{F_n} \approx 2.236\ldots = \sqrt{5}, \quad \text{and so} \quad F_n \approx \frac{\phi^n}{\sqrt{5}}.
\]
Rule: The $n$-th Fibonacci Number $F_n$ is the nearest whole number to $\frac{\phi^n}{\sqrt{5}}$.

**Example.** Find the 6-th and 13-th Fibonacci number.

$n = 6$. \[
\frac{\phi^6}{\sqrt{5}} = , \text{ so } F_6 =
\]

$n = 13$. \[
\frac{\phi^{13}}{\sqrt{5}} = , \text{ so } F_{13} =
\]

In fact, the exact formula is,

\[
F_n = \frac{1}{\sqrt{5}} \phi^n \pm \frac{1}{\sqrt{5} \phi^n}, \quad (+ \text{ for odd } n, \ - \text{ for even } n)
\]
Find $F_{100}$.

\[
\frac{\phi^{100}}{\sqrt{5}} = 354224848179261915075.00000000000000000000000056
\]

\[
F_{100} = 354224848179261915075
\]
Everyone calculate the following (round to three places):

$$\frac{1}{\phi} = \frac{1}{1.618} =$$

$$\phi^2 = 1.618^2 =$$
The Amazing Number $\phi = 1.61803398...$ (we’ll round to 3 places):

\[
\frac{1}{1.618} = .618 \quad \text{that is, } \frac{1}{\phi} = \phi - 1
\]
\[
1.618^2 = 2.618 = 1.618 + 1 \quad \text{that is, } \phi^2 = \phi + 1
\]
\[
1.618^3 = 4.236 = 2 \cdot 1.618 + 1 \quad \text{that is, } \phi^3 = 2\phi + 1
\]
\[
1.618^4 = 6.854 = 3 \cdot 1.618 + 2 \quad \text{that is, } \phi^4 = 3\phi + 2
\]
\[
1.618^5 = 11.089 = 5 \cdot 1.618 + 3 \quad \text{that is, } \phi^5 = 5\phi + 3
\]

\[
\phi^n = F_n\phi + F_{n-1}
\]
The Golden ratio is the unique positive real number satisfying the Golden Relation.
What is the exact value of \( \phi \)? From the Golden Relation, \[
\phi^2 - \phi - 1 = 0
\]

This is a quadratic equation (second degree): \( ax^2 + bx + c = 0 \). Quadratic Formula:

\[
\phi = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{1 + \sqrt{(-1)^2 - 4(-1)}}{2} = \frac{1 + \sqrt{5}}{2}.
\]

\[
\phi = \frac{1 + \sqrt{5}}{2} = 1.618033988749... \quad \text{irrational}
\]
• **Example.** Next, start with any two numbers and form a recursive sequence by adding consecutive numbers. See what the ratios approach this time.

Say we start with 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, ... 

<table>
<thead>
<tr>
<th>ratio</th>
<th>3</th>
<th>4</th>
<th>7</th>
<th>11</th>
<th>18</th>
<th>29</th>
<th>47</th>
<th>76</th>
<th>123</th>
</tr>
</thead>
<tbody>
<tr>
<td>value</td>
<td>3</td>
<td>1.33</td>
<td>1.75</td>
<td>1.57</td>
<td>1.64</td>
<td>1.61</td>
<td>1.62</td>
<td>1.617</td>
<td>1.618</td>
</tr>
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**Rule:** Starting with any two distinct positive numbers, and forming a sequence using the Fibonacci rule, the ratios of consecutive terms will always approach the Golden Ratio!

Recall the Fibonacci Rule: \( F_{n+1} = F_n + F_{n-1} \)
WHY? Let $A_n$ be a sequence satisfying the Fibonacci Rule:

$$A_{n+1} = A_n + A_{n-1}$$
**Golden Proportion**: Divide a line segment into two parts, such that the ratio of the longer part to the shorter part equals the ratio of the whole to the longer part. What is the ratio?
Example. Golden Rectangle: Form a rectangle such that when the rectangle is divided into a square and another rectangle, the smaller rectangle is similar (proportional) to the original rectangle. What is the ratio of the length to the width?
Start with a square $ABCD$. Mark the midpoint $J$ on a given edge $AB$. Draw an arc with compass point fixed at $J$ and passing through a vertex $C$ on the opposite edge. Mark the point $G$ where the arc meets the line $AB$.

- Note: Good approximations to the Golden Rectangle can be obtained using the Fibonacci Ratios.
Partitioning a Golden Rectangle into Squares

Diagram showing the partitioning of a golden rectangle into squares with labeled points A, B, C, D, E, F, G, and various smaller squares labeled with points H, I, J, K, L, M, N, O, P, Q.
Where is the eye of the spiral located?
• **Example.** The Pentagon and Pentagram.

The ratio of the edge of the inscribed star to the edge of the regular pentagon is $\phi$, the golden ratio.

The ratio of the longer part of an edge of the star to the shorter part is $\phi$. 
Construction of a Regular Pentagon
Golden Angle

Divide a circle into two arcs, so that the ratio of the longer arc to the smaller arc is the golden ratio.
Continued Fraction Expansions

• **Example.** Expand \( \pi = 3.141592653... \)

\[
\begin{align*}
3.141592... &= 3 + .141592... = 3 + \frac{1}{1/.141592...} = 3 + \frac{1}{7.062513...} \\
&= 3 + \frac{1}{7 + .062513...} = 3 + \frac{1}{7 + 1/.062513...} = 3 + \frac{1}{7 + 15.996594...} \\
&= 3 + \frac{1}{7 + \frac{1}{15 + .996594...}} = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1+...}}}
\end{align*}
\]

Every irrational number has a unique infinite continued fraction expansion.
Continued Fraction Expansion of $\phi$;

$1.6180339 = 1 + .6180339 = 1 + \frac{1}{1/.6180339} = 1 + \frac{1}{1.6180339}$

$= 1 + \frac{1}{1 + .6180339} = 1 + \frac{1}{1 + \frac{1}{1/.6180339}} = 1 + \frac{1}{1 + \frac{1}{1.6180339}}$

$= 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}}$

Another way to see this is: From the first line above we see that $\phi = 1 + \frac{1}{\phi}$, Now substitute this expression for $\phi$ into the right-hand side and keep repeating:

$\phi = 1 + \frac{1}{1 + \frac{1}{\phi}}, \ldots$
Convergents to the Continued Fraction expansion of $\phi$

$1, \quad 1 + \frac{1}{1} = 2, \quad 1 + \frac{1}{1 + \frac{1}{1}} = \frac{24}{24}$

The convergents to the continued fraction expansion of $\phi$ are