Cosmetic crossings and twisting operations in knots

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Joint work with Effie Kalfagianni
A **crossing change** in a knot diagram is when the overstrand and the understrand of one crossing in the diagram are switched.
Crossing disks and crossing circles

We want to be able to talk about crossing changes in a knot $K$ without restricting ourselves to any particular diagram of $K$. 
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Let $K$ be an oriented knot in $S^3$.

- A **crossing disc** for $K$ is an embedded disc $D \subset S^3$ such that $K$ intersects $\text{int}(D)$ twice with zero algebraic intersection number.
- $L = \partial D$ is a **crossing circle**.
A crossing change in a knot diagram is equivalent to performing $(\pm 1)$-Dehn surgery on the corresponding crossing circle.
Generalized crossing changes

Since a crossing change is equivalent to adding one full twist at $L$, we can define an order-$q$ generalized crossing change at $L$ to be $(-1/q)$-Dehn surgery on $L$, which is equivalent to adding $q$ full twists.
Nugatory and cosmetic crossing changes

Let $L$ be a crossing circle for $K$, and let $K_L(q)$ be the oriented knot obtained from $K$ via an order-$q$ crossing change at $L$. 
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- $L$ is **nugatory** if $L$ bounds a disk in $\overline{S^3 - \eta(K)}$.

- $L$ is **cosmetic** if $L$ is not nugatory and $K_L(q)$ is isotopic to $K$, i.e., there exists an orientation-preserving diffeomorphism $f : S^3 \rightarrow S^3$ with $f(K) = K_L(q)$.
Open questions

- **Nugatory crossing conjecture** (Problem 1.58 of Kirby’s list): Does there exist a knot $K$ which admits a cosmetic (traditional) crossing change? Conversely, if a crossing change on a knot $K$ yields a knot isotopic to $K$, must the crossing be nugatory?
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- **More generally**: Does there exist a knot $K$ which admits a cosmetic generalized crossing change of any order?
Known results

It has been shown that there are no cosmetic generalized crossing changes of any order for:

- Unknot (Scharleman, Thompson, 1989)
- 2-bridge knots (Torisu, 1999)
- Fibered knots (Kalfagianni, 2011)
- Genus-one algebraically non-slice knots (B., Friedl, Kalfagianni, Powell, 2012)

The question has also been reduced to the case of prime knots (Torisu), and reduced further to prime, hyperbolic knots if the order is greater than 5 (B.).
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Our main goal

Study potential cosmetic crossing changes in knots obtained via *twisting operations*.
Twist knots

Let $V \subset S^3$ be an unknotted solid torus and let $K \subset V$ be a knot such that the \textit{wrapping number} $\text{wr}(K, V) \geq 2$.
**Twist knots**

Let $V \subset S^3$ be an unknotted solid torus and let $K \subset V$ be a knot such that the *wrapping number* $\text{wr}(K, V) \geq 2$. Let $f : V \to V$ be given by an $n^{\text{th}}$ power Dehn twist. Then $K_{n,V} := f(K)$ is called a **twist knot** of $K$. 

![Diagram of a twist knot](image-url)
Theorem (B., Kalfagianni)

Let $K$ be a knot embedded in an unknotted solid torus $V$ such that $w(K, V) = \text{wr}(K, V) \geq 3$. If $K$ admits no cosmetic generalized crossing changes, then $\forall n \in \mathbb{Z}, \ K_n, V$ admits no cosmetic generalized crossing changes.
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Question

So what families of knots can this theorem be applied to? i.e. how can we find knots that

1. do not admit cosmetic generalized crossing changes and
2. are contained in a torus $V$ with $w(K, V) = \text{wr}(K, V) \geq 3$?
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Answer to (2)

Braids!!
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Answer to (2)

Braids!! More specifically, any closed $m$-braid $\hat{\beta}$ can be embedded in a solid torus $V$ with $w(K, V) = \text{wr}(K, V) \leq m$. 
Closed braids

So what closed braids are known not to admit cosmetic generalized crossing changes?

Fibered knots admit no cosmetic generalized crossing changes...
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Homogeneous braids

Recall that \( B_p \) is generated by \( \langle \sigma_i \rangle_{i=1}^{p-1} \). e.g. \( B_3 \) is generated by

\[
\sigma_1 = \quad \quad \quad \sigma_2 =
\]

\[
\downarrow \quad \downarrow \quad \downarrow
\]

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\[
\sigma_1 = \begin{array}{c}
\downarrow \\
\downarrow \\
\end{array} \\
\sigma_2 = \begin{array}{c}
\downarrow \\
\downarrow \\
\end{array}
\]

A braid is **homogeneous** if each time $\sigma_i$ appears, its exponent is always of the same sign.
Example

\[ \beta = \sigma_2^{-1} \sigma_4^{-1} \sigma_1 \sigma_3 \sigma_2^{-1} \sigma_4^{-2} \sigma_1 \] is homogeneous.
Theorem (Stallings, 1976)

If $\beta$ is a homogeneous braid, then the closure $\hat{\beta}$ is fibered.
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Notation
Let $K$ be a closed $m$-braid. Given $2 \leq p \leq m$ and $q \in \mathbb{Z}$, let $K_{p,q}$ denote a knot that is obtained by inserting $q$ full twists along $p$ strands of $K$. 
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Corollary 1
If $K = \hat{\beta}$ is a closed homogeneous $m$-braid the for all $3 \leq p \leq m$ and $q \in \mathbb{Z}$, $K_{p,q}$ admits no cosmetic generalized crossing changes of any order.
3-braids

**Theorem (Schreier, 1924)**

Suppose $K = \hat{\beta}$ for some $\beta \in B_3$. Then $\beta$ is conjugate to a braid in exactly one of the following forms, where $\Delta^2 = (\sigma_1\sigma_2)^3$, so $\Delta^{2k}$ is given by $k$ full twists.

1. $\Delta^{2k}\sigma_1^{p_1}\sigma_2^{-q_1}\cdots\sigma_1^{p_s}\sigma_2^{-q_s}$, where $k \in \mathbb{Z}$ and $p_i$, $q_i$, and $s$ are all positive integers
2. $\Delta^{2k}\sigma_1^p$ for some $k, p \in \mathbb{Z}$
3. $\Delta^{2k}\sigma_1\sigma_2$ for some $k \in \mathbb{Z}$
4. $\Delta^{2k}\sigma_1\sigma_2\sigma_1$ for some $k \in \mathbb{Z}$
5. $\Delta^{2k}\sigma_1\sigma_2\sigma_1\sigma_2$ for some $k \in \mathbb{Z}$.

Further, this form is unique up to cyclic permutation of the word following $\Delta^{2k}$.
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In particular, every knot which can be represented as the closure of a 3-braid is $\hat{\beta}_{3,k}$ for some homogeneous 3-braid $\beta$. 
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Corollary 2

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3-braids

In particular, every knot which can be represented as the closure of a 3-braid is $\hat{\beta}_{3,k}$ for some homogeneous 3-braid $\beta$.

**Corollary 2**

If $K$ can be represented as the closure of a 3-braid, then $K$ admits no cosmetic generalized crossing change of any order.

Thank you!