BILINEAR OPERATORS WITH HOMOGENEOUS SYMBOLS, SMOOTH MOLECULES, AND KATO-PONCE INEQUALITIES

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Abstract. We present a unifying approach to establish mapping properties for bilinear pseudodifferential operators with homogeneous symbols in the settings of function spaces that admit a discrete transform and molecular decompositions in the sense of Frazier and Jawerth. As an application, we obtain related Kato-Ponce inequalities.

1. Introduction and main results

As the main purpose of this note we present a unifying approach towards establishing mapping properties of the form

\[ \| T_\sigma(f, g) \|_Y \lesssim \| f \|_X \| g \|_{L^\infty} + \| f \|_{L^\infty} \| g \|_X, \]

where \( X \) and \( Y \) are function spaces admitting a molecular decomposition and a \( \varphi \)-transform in the sense of Frazier-Jawerth as introduced in [10, 11], and \( T_\sigma \) is a bilinear pseudodifferential operator given by

\[ T_\sigma(f, g)(x) := \int_{\mathbb{R}^{2n}} \sigma(x, \xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} \, d\xi \, d\eta \quad \forall x \in \mathbb{R}^n, \]

with a bilinear symbol \( \sigma \) in the class \( \dot{B}S^{m}_{1,1} \) for some \( m \in \mathbb{R} \), that is, \( \sigma \in C^\infty(\mathbb{R}^{3n} \setminus \{0\}) \) is such that for all multiindices \( \alpha, \beta, \gamma \in \mathbb{N}_0^n \) it holds

\[ \| \sigma \|_{\gamma, \alpha, \beta} := \sup_{(x, \xi, \eta) \in \mathbb{R}^{3n} \setminus \{0\}} |\partial_\xi^\gamma \partial_\eta^\alpha \sigma(x, \xi, \eta) (|\xi| + |\eta|)^{-m-|\gamma|+|\alpha+\beta|} < \infty. \]

When \( m = 0 \), the \( x \)-independent symbols in \( \dot{B}S^0_{1,1} \) constitute the well-known class of Coifman-Meyer bilinear multipliers. The bilinear forbidden class \( BS^0_{1,1} \) is defined as the family of symbols satisfying (1.2) with \( m = 0 \) and with \( |\xi| + |\eta| \) replaced by \( 1 + |\xi| + |\eta| \).

Note that if \( \sigma \) belongs to \( BS^0_{1,1} \) then \( \sigma = \sigma_1 + \sigma_2 \) where \( \sigma_1 \) is in \( \dot{B}S^0_{1,1} \) and \( \sigma_2 \) is a smoothing symbol supported in \( \{(x, \xi, \eta) : |\xi| + |\eta| \leq 1\} \). We refer the reader to the work of Coifman and Meyer in [7] and the references it contains for pioneering work related to such symbols.

As we will describe next, these two classes of symbols possess distinct essential features, and, as a noteworthy consequence of our Theorem 1.1 below, it will follow that they share various mapping properties of the form (1.1).
Coifman-Meyer bilinear multipliers can be realized as bilinear Calderón-Zygmund operators. As such, they inherit their mapping properties; for instance, Calderón-Zygmund operators are bounded in the settings of Lebesgue spaces, BMO, the Hardy space $H^1$ (Grafakos-Torres [15]), and in weighted Lebesgue spaces (Lerner et al. [21]).

On the other hand, the bilinear forbidden class $BS^{m}_{1,1}$ is known to produce bilinear pseudodifferential operators with a bilinear Calderón-Zygmund kernel, but, in general, they are not bilinear Calderón-Zygmund operators (Benyi-Torres [4]). In particular, they do not always possess mapping properties of the form $L^{p_1} \times L^{p_2} \to L^p$ with $1 < p_1, p_2 \leq \infty$ and $1/p_1 + 1/p_2 = 1/p > 0$. Mapping properties for bilinear pseudodifferential operators with symbols in $BS^{m}_{1,1}$ have been studied in Benyi [2] in the setting of Besov spaces, in Benyi-Torres [4] and Benyi-Nahmod-Torres [3] in the scale of Lebesgue-Sobolev spaces, and in Naibo [23] and Kocezuka-Tomita [20] in the context of Besov and Triebel-Lizorkin spaces.

As our main result, Theorem 1.1 below, we prove molecular estimates on $T_\sigma$, with $\sigma \in BS^{m}_{1,1}$, when one of its arguments is a fixed function and its other argument is a smooth molecule.

**Theorem 1.1.** Given $m \in \mathbb{R}$ and $\sigma \in BS^{m}_{1,1}$, there exist $\sigma^1, \sigma^2 \in BS^{m}_{1,1}$ with $T_\sigma = T_{\sigma^1} + T_{\sigma^2}$ and such that if $1 \leq r \leq \infty$, $0 < M < \infty$ and $\psi \in S(\mathbb{R}^n)$, with $\hat{\psi}$ supported in $\{ \xi \in \mathbb{R}^n : \frac{1}{2} < |\xi| < 2 \}$, it holds that

$$|\partial^\gamma T_{\sigma^1}(\psi_{\nu,k}, g)(x)| \lesssim \frac{2^{\frac{m}{2}} 2^{\nu(\gamma+1)} 2^{\frac{\nu k}{r}}}{(1 + |2^\nu x - k|)^M} \|g\|_{L^r} \quad \forall x \in \mathbb{R}^n$$

and

$$|\partial^\gamma T_{\sigma^2}(f, \psi_{\nu,k})(x)| \lesssim \frac{2^{\frac{m}{2}} 2^{\nu(\gamma+1)} 2^{\frac{\nu k}{r}}}{(1 + |2^\nu x - k|)^M} \|f\|_{L^r} \quad \forall x \in \mathbb{R}^n,$$

for every $\gamma \in \mathbb{N}_0^n$, $\nu \in \mathbb{Z}$, $k \in \mathbb{Z}^n$ and $f, g \in S(\mathbb{R}^n)$, and where $\psi_{\nu,k}(x) = 2^{\frac{\nu k}{r}} \psi(2^\nu x - k)$.

Here $S(\mathbb{R}^n)$ denotes the Schwartz class of smooth rapidly decreasing functions defined on $\mathbb{R}^n$; the notation $\lesssim$ means $\leq C$, where $C$ is a constant that may depend on some of the parameters used but not on the functions or variables involved.

1.1. **A sample of applications of Theorem 1.1.** In the case $r = \infty$, Theorem 1.1 implies that, up to uniform multiplicative constants, the functions $2^{-\nu m} T_{\sigma^1}(\psi_{\nu,k}, g)/\|g\|_{L^\infty}$ and $2^{-\nu m} T_{\sigma^2}(f, \psi_{\nu,k})/\|f\|_{L^\infty}$ can be regarded as smooth molecules, as introduced in [10, 11] in the settings of Besov and Triebel-Lizorkin spaces. Since smooth molecules also serve as building blocks for a variety of other function spaces, Theorem 1.1 will apply to such spaces as well.

As a concrete application, we will implement Theorem 1.1 in the scales of homogeneous Besov-type and Triebel-Lizorkin-type spaces. These spaces were introduced and studied in Sawano-Yang-Yuan [25] and Yang-Yuan [28, 29] as natural spaces that extend and unify the scales of homogeneous Besov spaces, homogeneous Triebel-Lizorkin spaces, and $Q$-spaces. The latter were introduced in Essén et al. [9] as a refinement of BMO functions. In addition, as proved in [25], the Besov-type and Triebel-Lizorkin-type spaces also contain or coincide with Besov-Morrey and Triebel-Lizorkin-Morrey spaces.

We refer the reader to Section 3 for detailed notation and precise definitions. In the following, $S_0(\mathbb{R}^n)$ denotes the closed subspace of functions in $S(\mathbb{R}^n)$ that have vanishing
moments of all orders; that is, \( f \in S_0(\mathbb{R}^n) \) if and only if \( f \in \mathcal{S}(\mathbb{R}^n) \) and \( \int_{\mathbb{R}^n} x^{\alpha} f(x) \, dx = 0 \) for all \( \alpha \in \mathbb{N}_0^n \). For \( 0 < p, q \leq \infty \), set

\[
\begin{align*}
    s_{p,q} := n \left( \min\{1,p,q\} - 1 \right) \quad \text{and} \quad s_p := n \left( \min\{1,p\} - 1 \right).
\end{align*}
\]

By means of Theorem 1.1 and molecular techniques, we obtain the following mapping properties in the scales of homogeneous Besov-type and Triebel-Lizorkin-type spaces.

**Theorem 1.2.** Let \( m \in \mathbb{R} \) and \( \sigma \in \dot{B}^m_{1,1} \). If \( 0 < p, q \leq \infty \), \( s_p < s < \infty \) and \( 0 \leq \tau < \frac{1}{p} + \frac{s - s_p}{n} \), it holds that

\[
\| T_\sigma(f,g) \|_{\dot{B}^{s+m,\tau}_{p,q}} \lesssim \| f \|_{\dot{B}^{s+m,\tau}_{p,q}} \| g \|_{L^\infty} + \| f \|_{L^\infty} \| g \|_{\dot{B}^{s+m,\tau}_{p,q}} \quad \forall f, g \in S_0(\mathbb{R}^n).
\]

If \( 0 < p < \infty \), \( 0 < q \leq \infty \), \( s_{p,q} < s < \infty \) and \( 0 \leq \tau < \frac{1}{p} + \frac{s - s_{p,q}}{n} \), it holds that

\[
\| T_\sigma(f,g) \|_{\dot{F}^{s,\tau}_{p,q}} \lesssim \| f \|_{\dot{F}^{s+m,\tau}_{p,q}} \| g \|_{L^\infty} + \| f \|_{L^\infty} \| g \|_{\dot{F}^{s+m,\tau}_{p,q}} \quad \forall f, g \in S_0(\mathbb{R}^n).
\]

Theorem 1.2 can be considered as a bilinear counterpart to Grafakos-Torres [14, Theorems 1.1 and 1.2] (see also [27]), where boundedness properties in homogeneous Besov and Triebel-Lizorkin spaces were addressed for linear pseudodifferential operators with symbols in the class \( S_{1,1} \), the linear analog to \( \dot{B}^m_{1,1} \). In turn, the (linear) results in [14] were extended to the setting of Besov-type and Triebel-Lizorkin-type spaces in [25, Theorem 1.5]. We refer the reader to Hart-Torres-Wu [17] where very different techniques are used to obtain estimates in the spirit of those in Theorem 1.2 in the setting of Sobolev spaces for operators with \( x \)-independent symbols and a limited amount of regularity.

In Remark 4.1 we address Theorem 1.2 in the cases corresponding to \( s \leq s_p \) and \( s \leq s_{p,q} \) and show that analogous estimates are obtained, with a slightly different range for the parameter \( \tau \), if a number of cancellation conditions are imposed on the first adjoint of \( T_{\sigma_1} \) and on the second adjoint of \( T_{\sigma_2} \), where \( \sigma_1 \) and \( \sigma_2 \) are as in Theorem 1.1. In Remark 4.2 we give a version of Theorem 1.2 involving the \( L^r \) norms of \( f \) and \( g \) instead of their \( L^\infty \) norms.

The next corollary of Theorem 1.2 follows from the realization of \( Q \)-spaces as special cases of Triebel-Lizorkin-type spaces (see Section 3.1.1).

**Corollary 1.3.** Let \( s, s + m \in (0,1) \) and \( \sigma \in \dot{B}^m_{1,1} \). If \( 1 \leq q \leq p \leq \infty \) and \( q \neq \infty \), it holds that

\[
\| T_\sigma(f,g) \|_{\dot{Q}^{s,q}_{p,q}} \lesssim \| f \|_{\dot{Q}^{s+m,q}_{p,q}} \| g \|_{L^\infty} + \| f \|_{L^\infty} \| g \|_{\dot{Q}^{s+m,q}_{p,q}} \quad \forall f, g \in S_0(\mathbb{R}^n).
\]

### 1.2. Applications to Kato-Ponce inequalities

As a consequence of Theorem 1.1 in the case \( \sigma \equiv 1 \), given a function space \( X \) that admits a molecular representation and a \( \varphi \)-transform, we obtain the following fractional Leibniz rule or Kato-Ponce inequality:

\[
\| fg \|_X \lesssim \| f \|_X \| g \|_{L^\infty} + \| f \|_{L^\infty} \| g \|_X.
\]

Inequalities of the form (1.4) were proved by Kato-Ponce [18] in the case where \( X \) is the Sobolev space \( W^{s,p}(\mathbb{R}^n) \), with \( 1 < p < \infty \) and \( 0 < s < \infty \), in relation to Cauchy problems for the Euler and Navier-Stokes equations; prior work due to Strichartz [26] treats the range \( n/p < s < 1 \), while the case of \( s \in \mathbb{N} \) can be obtained from the Leibniz rule and the Gagliardo-Nirenberg inequality. Later on, Gulisashvili-Kon [16] showed (1.4) for the homogeneous space \( X = \dot{W}^{s,p}(\mathbb{R}^n) \), for the same range of parameters, in connection with the study of smoothing properties of Schrödinger semigroups. The estimates (1.4) also hold true in the settings of Besov and Triebel-Lizorkin spaces and have applications to partial differential equations (see, for instance, Bahouri-Chemin-Danchin [1], Chae [5], Runst-Sickel [24] and
the references they contain). In particular, all such estimates imply that $X \cap L^\infty(\mathbb{R}^n)$ is an algebra under pointwise multiplication. Closely related versions to (1.4) were given by Christ-Weinstein [6] and Kato-Ponce-Vega [19], in the contexts of Korteweg-de Vries equations, and by Gulisashvili-Kon [16]. Extensions to the cases of indices below one appear in Grafakos-Oh [13] and Muscalu-Schlag [22], and versions in weighted and variable exponent space settings were proved in Cruz-Uribe-Naibo [8].

In particular, in the scales of Besov-type and Triebel-Lizorkin-type spaces, Theorem 1.2 yields the following new Kato-Ponce inequalities.

**Corollary 1.4.** If $0 < p, q \leq \infty$, $s_p < s < \infty$ and $0 \leq \tau < \frac{1}{p} + \frac{s_p}{n}$, it holds that

$$
\|fg\|_{\dot{B}^{s,\tau}_{p,q}} \lesssim \|f\|_{\dot{B}^{s,\tau}_{p,q}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{B}^{s,\tau}_{p,q}} \quad \forall f, g \in S_0(\mathbb{R}^n).
$$

If $0 < p < \infty$, $0 < q \leq \infty$, $s_{p,q} < s < \infty$ and $0 \leq \tau < \frac{1}{p} + \frac{s_{p,q}}{n}$, it holds that

$$
\|fg\|_{\dot{F}^{s,\tau}_{p,q}} \lesssim \|f\|_{\dot{F}^{s,\tau}_{p,q}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^{s,\tau}_{p,q}} \quad \forall f, g \in S_0(\mathbb{R}^n).
$$

If $0 < s < 1$, $1 \leq q \leq \infty$ and $p \neq \infty$, it holds that

$$
\|fg\|_{Q_p^{s,q}} \lesssim \|f\|_{Q_p^{s,q}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{Q_p^{s,q}} \quad \forall f, g \in S_0(\mathbb{R}^n).
$$

The article is organized as follows. In Section 2 we prove Theorem 1.1. Section 3 contains the definitions of Besov-type and Triebel-Lizorkin-type spaces, smooth molecules, and the $\varphi$-transform. The proof of Theorem 1.2 and several closing remarks are given in Section 4.

## 2. Proof of Theorem 1.1

Our first step towards the proof of Theorem 1.1 will be obtaining a representation of a bilinear pseudodifferential operator with a symbol in $\dot{B}S_{1,1}^m$ as a superposition of paraproduct-like operators. Such representations can be traced back to the pioneering work of Coifman and Meyer; Lemma 2.1 gives a version of a decomposition suited for our purposes, and its proof follows ideas inspired from [7, pp.154-155]. We then state and prove Lemma 2.2, which procures a formula for the derivatives of the building blocks, appropriately evaluated, given by Lemma 2.1. We close this section with the proof of Theorem 1.1.

The Fourier transform of a tempered distribution $f \in S'(\mathbb{R}^n)$ will be denoted by $\hat{f}$; in particular, we use the formula $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x \cdot \xi} dx$ for $f \in S(\mathbb{R}^n)$.

Let $\theta$ be a real-valued infinitely differentiable function supported on $(-2, 2)$ and such that $\theta(t) + \theta(1/t) = 1$ for every $t > 0$. For $\sigma \in \dot{B}S_{1,1}^m$, $m \in \mathbb{R}$, define

$$
\sigma^1(x, \xi, \eta) := \sigma(x, \xi, \eta)\theta \left( \frac{|\eta|}{m} \right) \quad \text{and} \quad \sigma^2(x, \xi, \eta) := \sigma(x, \xi, \eta)\theta \left( \frac{|\xi|}{m} \right) \quad \forall x, \xi, \eta \in \mathbb{R}^n.
$$

Simple computations show that $\sigma^1, \sigma^2 \in \dot{B}S_{1,1}^m$ with

$$
\|\sigma^d\|_{\gamma,\alpha,\beta} \lesssim \sup_{\alpha + \beta \leq \gamma} \|\sigma\|_{\gamma,\alpha,\beta} \quad \text{for } \gamma, \alpha, \beta \in \mathbb{N}_0^n \text{ and } d = 1, 2,
$$

where the implicit constant depends only on $\gamma, \alpha, \beta$ and $\theta$, and we have

$$
T_{\sigma}(f, g) = T_{\sigma^1}(f, g) + T_{\sigma^2}(f, g), \quad \forall f, g \in S_0(\mathbb{R}^n).
$$

Endowing $S_0(\mathbb{R}^n)$ with the topology inherited from $S(\mathbb{R}^n)$, a standard argument using integration by parts allows to conclude that $T_{\sigma^1}$ is continuous from $S_0(\mathbb{R}^n) \times S(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$ and $T_{\sigma^2}$ is continuous from $S(\mathbb{R}^n) \times S_0(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$. Let $\Psi, \Phi \in S(\mathbb{R}^n)$ be such that $\hat{\Psi}$ and
\( \hat{\Phi} \) are real-valued, \( \text{supp}(\hat{\Psi}) \subset \{ \xi : \frac{1}{2} < |\xi| < 2 \} \), \( \sum_{j \in \mathbb{Z}} |\hat{\Psi}(2^{-j}\xi)|^2 = 1 \) for every \( \xi \neq 0 \), \( \hat{\Phi} \equiv 1 \) for \( |\xi| \leq 4 \) and \( \hat{\Phi} \equiv 0 \) for \( |\xi| > 10 \).

**Lemma 2.1.** Let \( \sigma \in \dot{B}S_{1,1}^m \). With the notation introduced above and given \( N > n \), there exist sequences of functions \( \{m_j^1(x,u,v)\}_{j \in \mathbb{Z}} \) and \( \{m_j^2(x,u,v)\}_{j \in \mathbb{Z}} \) defined for \( x,u,v \in \mathbb{R}^n \) such that if \( \gamma \in \mathbb{N}_0^n \), then

\[
\tag{2.5} \sup_{u,v \in \mathbb{R}^n} |\partial^\gamma m_j^0(x,u,v)| \lesssim 2^{j(m+|\gamma|)}, \quad \forall j \in \mathbb{Z}, d = 1, 2,
\]

and, if \( f \in \mathcal{S}_0(\mathbb{R}^n) \), \( g \in \mathcal{S}(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \), it holds that

\[
\tag{2.6} T_{\sigma^1}(f,g)(x) = \int_{\mathbb{R}^{2n}} \sum_{j \in \mathbb{Z}} m_j^1(x,u,v) \Delta_j^uf(x) S_j^ug(x) \frac{dudv}{(1+|u|^2+|v|^2)^N}
\]

and

\[
\tag{2.7} T_{\sigma^2}(g,f)(x) = \int_{\mathbb{R}^{2n}} \sum_{j \in \mathbb{Z}} m_j^2(x,u,v) S_j^ug(x) \Delta_j^uf(x) \frac{dudv}{(1+|u|^2+|v|^2)^N},
\]

where \( \Delta_j^u = \hat{\Psi}(2^{-j}\xi) \hat{f}(\xi) \) with \( \hat{\Psi}(\xi) := \Psi(x + u) \) and \( \hat{\Delta}_j^u = \hat{\Phi}(2^{-j}\xi) \hat{g}(\xi) \) with \( \hat{\Phi}(x) := \Phi(x + v) \).

**Proof.** We will prove (2.6), with the proof of (2.7) following analogously. Since the support of \( |\hat{\Psi}(2^{-j}\xi)|^2 \sigma^1(x,\xi,\eta) \) is contained in \( \{ (x,\xi,\eta) : |\eta| \leq 2|\xi| \text{ and } 2^{j-1} < |\xi| < 2^{j+1} \} \subset \{ (x,\xi,\eta) : |\eta| \leq 2^{j+2} \} \) and \( \hat{\Phi}(2^{-j}\eta) \equiv 1 \) for \( |\eta| \leq 2^{j+2} \), we have

\[
|\hat{\Psi}(2^{-j}\xi)|^2 \sigma^1(x,\xi,\eta) = |\hat{\Phi}(2^{-j}\eta)|^2 |\hat{\Psi}(2^{-j}\xi)|^2 \sigma^1(x,\xi,\eta) \quad \forall x,\xi,\eta \in \mathbb{R}^n, j \in \mathbb{Z}.
\]

From this, the fact that \( \sum_{j \in \mathbb{Z}} |\hat{\Psi}(2^{-j}\xi)|^2 = 1 \) for \( \xi \neq 0 \) and Fubini’s theorem, it follows that if \( f \in \mathcal{S}_0(\mathbb{R}^n) \) and \( g \in \mathcal{S}(\mathbb{R}^n) \), then

\[
\tag{2.8} T_{\sigma^1}(f,g)(x) = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^{2n}} \sigma^1_j(x,2^{-j}\xi,2^{-j}\eta) \hat{\Psi}(2^{-j}\xi) \hat{\Phi}(2^{-j}\eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i(x+\xi+\eta)} d\xi d\eta,
\]

where \( \sigma^1_j(x,\xi,\eta) := \hat{\Psi}(\xi) \hat{\Phi}(\eta) \sigma^1(x,2^j\xi,2^j\eta) \).

Given multiindices \( \gamma, \alpha, \beta \in \mathbb{N}_0^n \), the Leibniz rule implies that \( \partial^\gamma \partial^\alpha \partial^\beta \sigma^1 \) can be written as a linear combination of terms of the form

\[
\tag{2.9} \partial^\alpha \hat{\Psi}(\xi) \partial^\beta \hat{\Phi}(\eta) (\partial^\gamma \hat{\xi}) \hat{\sigma}^1(x,2^j\xi,2^j\eta) 2^{|\alpha|+|\beta|}
\]

\[
\alpha_1 + \alpha_2 = \alpha, \beta_1 + \beta_2 = \beta.
\]

Since \( \sigma^1 \in \dot{B}S_{1,1}^m \), the absolute value of each term (2.9) can be bounded by a multiple of

\[
|\partial^\alpha \hat{\Psi}(\xi) \partial^\beta \hat{\Phi}(\eta)| 2^{|\alpha|+|\beta|} (|2^j\xi| + |2^j\eta|)^{m+|\gamma|} \lesssim 2^j (m+|\gamma|) \forall x,\xi,\eta \in \mathbb{R}^n,
\]

where we have used that \( \partial^\alpha \hat{\Psi}(\xi) \partial^\beta \hat{\Phi}(\eta) \) is supported in \( \{(\xi,\eta) : \frac{1}{2} < |\xi| + |\eta| < 12 \} \), and the explicit constant is independent of \( j \).

Define \( m_j^1(x,u,v) := (1+|u|^2+|v|^2)^N \sigma^1_j(x,\xi,\cdot)(u,v) \); by the above we have

\[
|\partial^\gamma m_j^1(x,u,v)| = (1+|u|^2+|v|^2)^N \left| \int_{\mathbb{R}^{2n}} \partial^\gamma \sigma^1_j(x,\xi,\eta) \frac{(1-\Delta_{\xi,\eta})^N e^{-2\pi i(u-\xi+v-\eta)}}{(1+4\pi^2|u|^2+4\pi^2|v|^2)^N} d\xi d\eta \right|
\]

\[
\sim \left| \int_{\frac{1}{2} < |\xi| + |\eta| < 12} (1-\Delta_{\xi,\eta})^N (\partial^\gamma \sigma^1_j)(x,\xi,\eta)e^{-2\pi i(u-\xi+v-\eta)} d\xi d\eta \right| \lesssim 2^j (m+|\gamma|).
\]
Finally, using that
\[
\sigma_j^1(x, 2^{-j}\xi, 2^{-j}\eta) = \int_{\mathbb{R}^n} m_j^1(x, u, v) e^{2\pi i (u \cdot 2^{-j}\xi + v \cdot 2^{-j}\eta)} \frac{dudv}{(1 + |u|^2 + |v|^2)^N}
\]
in (2.8), after interchanging summation and integral signs justified by Fubini’s theorem, we get (2.6).

For each \( u, v \in \mathbb{R}^n \), set
\[
\sigma_{u,v}^1(x, \xi, \eta) := \sum_{j \in \mathbb{Z}} m_j^1(x, u, v) \hat{\psi}^u(2^{-j}\xi) \hat{\phi}^v(2^{-j}\eta),
\]
then \( T_{\sigma_{u,v}^1}(f, g)(x) = \sum_{j \in \mathbb{Z}} m_j^1(x, u, v) \Delta_j^u f(x) S_j^v g(x) \). Similarly define \( \sigma_{u,v}^2 \). In our next lemma we look at derivatives of \( T_{\sigma_{u,v}^1}(\psi_{\nu,k}, g) \) and \( T_{\sigma_{u,v}^2}(f, \psi_{\nu,k}) \).

**Lemma 2.2.** If \( \gamma \in \mathbb{N}_0^n, \nu \in \mathbb{Z}, k \in \mathbb{Z}^n, u, v \in \mathbb{R}^n, g \in S(\mathbb{R}^n) \) and \( \psi \in S(\mathbb{R}^n) \) is such that \( \text{supp}(\hat{\psi}) \subset \{ \xi \in \mathbb{R}^n : \frac{1}{2} < |\xi| < 2 \} \), then

\[
\partial^\gamma T_{\sigma_{u,v}^1}(\psi_{\nu,k}, g)(x) = 2^{n\nu} \sum_{j=\nu-1}^{\nu+1} C_{\gamma_1, \gamma_2, \gamma_3} 2^{\nu|\gamma - \gamma_1|} \partial_{x_1}^{\gamma_1} m_j^1(x, u, v) \times (\Phi_{\nu-\gamma}^2 \ast g(2^{-\nu}))(2^\nu x + 2^{-\nu-j} v) \Psi_{\nu-k}^3(2^\nu x - k + 2^{-\nu-j} u),
\]
where \( \Phi_{\nu-\gamma}^2, \Psi_{\nu-k}^3 \in S(\mathbb{R}^n) \) are independent of \( g \) and \( \psi_{\nu,k}(x) = 2^{\frac{\mu-1}{2}} \psi(2^\nu x - k) \). An analogous formula holds for \( \partial^\gamma T_{\sigma_{u,v}^2}(f, \psi_{\nu,k}) \) with \( f \in S(\mathbb{R}^n) \).

**Proof.** In view of the supports of \( \hat{\psi} \) and \( \hat{\Psi} \), the supports of \( \hat{\psi}(2^{-\nu} \cdot) \) and \( \hat{\Psi}(2^{-\nu} \cdot) \) only intersect if \( \nu - 1 \leq j \leq \nu + 1 \). We then have

\[
T_{\sigma_{u,v}^1}(\psi_{\nu,k}, g)(x) = \sum_{j=\nu-1}^{\nu+1} m_j^1(x, u, v) \int_{\mathbb{R}^n} \hat{\psi}_{\nu,k}(\xi) \hat{\phi}^v(2^{-j}\eta) g(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta
\]

\[
= \sum_{j=\nu-1}^{\nu+1} m_j^1(x, u, v) 2^{-n\nu} \int_{\mathbb{R}^n} \hat{\psi}_{\nu,k}(\xi) \hat{\phi}^v(2^{-j}\eta) e^{-2\pi i 2^{-\nu-j} \nu \cdot \xi} \hat{\psi}(2^{-\nu}\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta
\]

\[
= \sum_{j=\nu-1}^{\nu+1} 2^{n\nu} m_j^1(x, u, v) \left( \int_{\mathbb{R}^n} 2^n \hat{\phi}^v(2^{-\nu}\eta) e^{2\pi i 2^{-\nu} x \cdot \eta} d\eta \right) \left( \int_{\mathbb{R}^n} \hat{\psi}(2^{-\nu}\xi) e^{2\pi i (2^{\nu} x - k) \cdot \xi} d\xi \right)
\]

Denoting

\[
F_j(x) := m_j^1(x, u, v) \left( \int_{\mathbb{R}^n} 2^n \hat{\phi}^v(2^{-\nu}\eta) e^{2\pi i 2^{-\nu} x \cdot \eta} d\eta \right) \left( \int_{\mathbb{R}^n} \hat{\psi}(2^{-\nu}\xi) e^{2\pi i (2^{\nu} x - k) \cdot \xi} d\xi \right)
\]
and given a multi-index $\gamma \in \mathbb{N}_0^n$, we have

$$
\partial^\gamma F_j(x) = \sum_{\gamma_1 + \gamma_2 + \gamma_3 = \gamma} C_{\gamma_1, \gamma_2, \gamma_3} \partial^{\gamma_1} m_j^{\gamma_2}(x, u, v) \\
\times \left( \int_{\mathbb{R}^n} 2^{m_\nu} \hat{g}(2^\nu \eta) 2^{\nu|\gamma_2|} \eta \hat{\phi}(2^\nu \eta) e^{2\pi i 2^\nu x \cdot \eta} \, d\eta \right) \left( \int_{\mathbb{R}^n} 2^{\nu|\gamma_3|} \xi \hat{\psi}(2^\nu \xi) e^{2\pi i (2^\nu x - k) \cdot \xi} \, d\xi \right)
$$

$$
= \sum_{\gamma_1 + \gamma_2 + \gamma_3 = \gamma} C_{\gamma_1, \gamma_2, \gamma_3} 2^{\nu|\gamma - \gamma_1|} \partial^{\gamma_1} m_j^{\gamma_2}(x, u, v) \\
\times \left( \int_{\mathbb{R}^n} 2^{m_\nu} \hat{g}(2^\nu \eta) \eta \hat{\phi}(2^\nu \eta) e^{2\pi i (2^\nu x - k + 2^\nu u) \cdot \eta} \, d\eta \right) \left( \int_{\mathbb{R}^n} \xi \hat{\psi}(2^\nu \xi) e^{2\pi i (2^\nu x - k) \cdot \xi} \, d\xi \right)
$$

where $\hat{\phi}^{\nu-j} := \eta \hat{\phi}(2^\nu \eta)$ and $\hat{\psi}^{\nu-j} := \xi \hat{\psi}(2^\nu \xi)$. Since $\partial^\gamma T_{\sigma_{1,\nu}}(\psi_{\nu,k}, g)(x) = \sum_{j=\nu-1}^{\nu+1} 2^{m_\gamma} \partial^\gamma F_j(x)$, we get the desired result.

**Proof of Theorem 1.1.** Let $\sigma \in BS_{1,1}^m$, $1 \leq r \leq \infty$, $0 < M < \infty$, $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{\psi}$ is supported in $\{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$ and $g \in \mathcal{S}(\mathbb{R}^n)$. With the notation used above, Lemma 2.2 and (2.5) imply

$$
\left| \partial^\gamma T_{\sigma_{1,\nu}}(\psi_{\nu,k}, g)(x) \right| \lesssim 2^{\nu M} 2^{\nu(m+|\gamma|)} \sum_{j=\nu-1}^{\nu+1} \left\| \Phi^{\nu-j} * g(2^\nu \cdot) \right\|_{L^\infty} \left| \Psi^{\nu-j}_M (2^\nu x - k + 2^\nu u) \right|
$$

$$
\lesssim 2^{\nu M} 2^{\nu(m+|\gamma|)} \sum_{j=\nu-1}^{\nu+1} \left\| \Phi^{\nu-j} \right\|_{L^{\nu'}} \left\| g(2^\nu \cdot) \right\|_{L^r} \frac{(1 + |2^\nu u|)^M}{(1 + |2^\nu x - k|)^M}
$$

where in the second inequality we have used that $\Psi^{\nu-j}_M \in \mathcal{S}(\mathbb{R}^n)$. Since

$$
T_{\sigma_1}(f, g)(x) = \int_{\mathbb{R}^{2n}} T_{\sigma_{1,\nu}}(f, g)(x) \, du dv \frac{du dv}{(1 + |u|^M + |v|^M)^N},
$$

by choosing $N$ sufficiently large so that $\int_{\mathbb{R}^{2n}} \frac{(1 + |u|^M + |v|^M)^N}{(1 + |2^\nu x - k|^M)^M} \, du dv < \infty$, we obtain the desired estimate for $\partial^\gamma T_{\sigma_1}(\psi_{\nu,k}, g)(x)$. An analogous reasoning leads to the estimate for $\partial^\gamma T_{\sigma_2}(f, \psi_{\nu,k})(x)$.

## 3. Function spaces

We recall that $\mathcal{S}_0(\mathbb{R}^n)$ denotes the closed subspace of functions in $\mathcal{S}(\mathbb{R}^n)$ that have vanishing moments of all orders and we endow $\mathcal{S}_0(\mathbb{R}^n)$ with the topology inherited from $\mathcal{S}(\mathbb{R}^n)$. The dual space of $\mathcal{S}_0(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$, can be identified with the space of tempered distributions modulo polynomials, $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$.

Let $\mathcal{D}$ be the collection of dyadic cubes in $\mathbb{R}^n$. That is, $\mathcal{D} := \{Q_{\nu,k}\}_{\nu \in \mathbb{Z}, k \in \mathbb{Z}^n}$ where

$$
Q_{\nu,k} := \{ x \in \mathbb{R}^n : k_j \leq 2^\nu x_j < k_j + 1, j = 1, \ldots, n \}.
$$
We denote the edge length of $Q_{\nu,k}$ by $l(Q_{\nu,k})$ and set $x_Q = x_{\nu,k} := 2^{-\nu}k$ where $Q = Q_{\nu,k}$.

We will consider functions $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ such that

(3.10) $\text{supp}(\widehat{\varphi}), \text{supp}(\widehat{\psi}) \subset \{ \xi \in \mathbb{R}^n : \frac{1}{2} < |\xi| < 2 \}$,

(3.11) $|\widehat{\varphi}(\xi)|, |\widehat{\psi}(\xi)| > c$ for all $\xi$ such that $\frac{3}{5} < |\xi| < \frac{5}{3}$ and some $c > 0$,

(3.12) $\sum_{j \in \mathbb{Z}} \widehat{\varphi}(2^{-j}\xi)\widehat{\psi}(2^{-j}\xi) = 1$ for $\xi \neq 0$.

See [12, Lemma 6.9] for a construction of $\psi$ given that $\varphi$ satisfies (3.10) and (3.11).

If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies (3.10) and (3.11), $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, we recall that $\varphi_{\nu,k}$ denotes the $L^2$-normalized function $\varphi_{\nu,k}(x) = 2^{n\nu} \varphi(2^n x - k) = 2^{\nu_n} \varphi(2^n(x - x_{\nu,k}))$. If $\psi \in \mathcal{S}(\mathbb{R}^n)$ verifies (3.10), (3.11) and (3.12), then it follows that

$$f = \sum_{\nu \in \mathbb{Z}, k \in \mathbb{Z}^n} \langle f, \varphi_{\nu,k} \rangle \psi_{\nu,k},$$

where the series converges for $f \in L^2(\mathbb{R}^n)$ in the topology of $L^2(\mathbb{R}^n)$, for $f \in \mathcal{S}_0(\mathbb{R}^n)$ in the topology of $\mathcal{S}(\mathbb{R}^n)$ and for $f \in \mathcal{S}'(\mathbb{R}^n)$ in $\mathcal{S}'(\mathbb{R}^n)$ modulo polynomials (see [10, 11] for details).

### 3.1. Homogeneous Besov-type and Triebel-Lizorkin-type spaces

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy conditions (3.10) and (3.11), and set $\varphi_j(x) := 2^{jn} \varphi(2^j x)$ for $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$. Fix $s, \tau \in \mathbb{R}$ and $0 < q \leq \infty$. For $0 < p \leq \infty$, the Besov-type space $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ is defined as the set of all $f \in \mathcal{S}'_0(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}^{s,\tau}_{p,q}} := \sup_{P \in \mathcal{D}} \frac{1}{|P|^\tau} \left\{ \sum_{j = -\log_2(l(P))}^{\infty} \left[ \int_P \left(2^j |\varphi_j * f(x)|^p\right)^{q/p} \right]^{1/q} \right\} < \infty.$$

For $0 < p < \infty$, the Triebel-Lizorkin-type space $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ is defined as the set of all $f \in \mathcal{S}'_0(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^{s,\tau}_{p,q}} := \sup_{P \in \mathcal{D}} \frac{1}{|P|^\tau} \left\{ \int_P \left[ \sum_{j = -\log_2(l(P))}^{\infty} (2^j |\varphi_j * f(x)|^q)^{1/p} \right] dx \right\} < \infty.$$

These spaces are independent of the choice of $\varphi$ (see [29, Corollary 3.1]). As in [29], we will use $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ to denote either $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ or $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$, excluding $p = \infty$ in the latter case.

#### 3.1.1. Special cases of $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$

We refer the reader to [28, Section 3] and [29, Proposition 3.1] regarding the following statements.

(i) If $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and $-\infty < \tau < 0$, then $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ equals the equivalence class of all polynomials on $\mathbb{R}^n$; if $0 \leq \tau < \infty$, they are quasi-Banach spaces and contain $\mathcal{S}_0(\mathbb{R}^n)$.

(ii) If $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and $\tau = 0$, then $\dot{B}^{s,0}_{p,q}(\mathbb{R}^n)$ coincides with the homogeneous Besov space $\dot{B}^{s}_{p,q}(\mathbb{R}^n)$, with equivalent norms.

(iii) If $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $\tau = 0$, then $\dot{F}^{s,0}_{p,q}(\mathbb{R}^n)$ coincides with the homogeneous Triebel-Lizorkin space $\dot{F}^{s}_{p,q}(\mathbb{R}^n)$, with equivalent norms. In turn, $\dot{F}^{s}_{p,\infty}(\mathbb{R}^n)$ coincides with the Sobolev space $W^{s,p}(\mathbb{R}^n)$ for $1 < p < \infty$ and $0 < s < \infty$, with equivalent norms.
(iv) If $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$, then $\dot{F}_{p,q}^{s,1}(\mathbb{R}^n)$ coincides with the homogeneous Triebel-Lizorkin space $\dot{F}_{\infty,q}^{s}(\mathbb{R}^n)$, with equivalent norms. In particular, $F_{0,2}^{0,1}(\mathbb{R}^n) = BMO(\mathbb{R}^n)$, with equivalent norms.

(v) If $0 < p \leq \infty$, $1 \leq q < \infty$ and $0 < s < 1$, then $\dot{F}_{p,q}^{s,1-\frac{1}{p}}(\mathbb{R}^n)$ coincides with the $Q$-space $Q_p^{s,q}(\mathbb{R}^n)$, with equivalent norms. Here $f \in Q_p^{s,q}(\mathbb{R}^n)$ if and only if $f \in S'_0(\mathbb{R}^n)$ with $f(x) - f(y)$ measurable on $\mathbb{R}^n \times \mathbb{R}^n$ and

$$
\|f\|_{Q_p^{s,q}(\mathbb{R}^n)} := \sup_I |I|^{1/p-1/q} \left\{ \int_I \int_I \frac{|f(x) - f(y)|^q}{|x-y|^{n+qs}} \, dy \, dx \right\}^{1/q} < \infty,
$$

where $I$ ranges over all cubes of $\mathbb{R}^n$ with dyadic edge lengths. In particular, $Q_s(\mathbb{R}^n) := Q_{n,s}^{n,2}(\mathbb{R}^n) = \dot{F}_{2,2}^{s,1-\frac{1}{2}}(\mathbb{R}^n)$. For $0 < s < 1$ if $n \geq 2$, or for $0 < s \leq \frac{1}{2}$ if $n = 1$, the spaces $Q_s(\mathbb{R}^n)$ constitute a decreasing family of nontrivial subspaces of $BMO$, see [9].

(vi) Further special cases of the spaces $\dot{A}_{p,q}^{s,\gamma}(\mathbb{R}^n)$ involving homogeneous Besov-Morrey and Triebel-Lizorkin-Morrey spaces can be found in [25, Theorem 1.1].

3.1.2. Molecules. Based on the pioneering work from [10, 11], it was proved in [29, Theorem 3.1] that the spaces $\dot{A}_{p,q}^{s,\gamma}(\mathbb{R}^n)$ can be characterized in terms of the so-called $\varphi$-transform defined by $S_\varphi(f) = \{ (f, \varphi_{\nu,k}) \}_{\nu,k}$ for $f \in S'_0(\mathbb{R}^n)$, where $\varphi \in S(\mathbb{R}^n)$ satisfies (3.10) and (3.11). More precisely, if $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and $0 \leq \tau < \infty$, then

$$
\|f\|_{B_{p,q}^{s,\gamma}} \sim \|\{ (f, \varphi_{\nu,k}) \}_{\nu,k}\|_{\dot{B}_{p,q}^{s,\gamma}} \quad \text{and} \quad \|f\|_{F_{p,q}^{s,\gamma}} \sim \|\{ (f, \varphi_{\nu,k}) \}_{\nu,k}\|_{\dot{F}_{p,q}^{s,\gamma}},
$$

where $\dot{B}_{p,q}^{s,\gamma}$ and $\dot{F}_{p,q}^{s,\gamma}$ refer to the following spaces of sequences: For $0 < p \leq \infty$, the space $\dot{B}_{p,q}^{s,\gamma}(\mathbb{R}^n)$ is defined as the collection of all sequences $t = \{ t_Q \}_{Q \in \mathcal{D}} \subset \mathbb{C}$, indexed by the dyadic cubes, such that

$$
\|t\|_{\dot{B}_{p,q}^{s,\gamma}} := \sup_{P \in \mathcal{D}} \frac{1}{|P|^\tau} \left\{ \sum_{j = -\log_2(t(P))}^{\infty} \int_P \left( \sum_{t(Q) = 2^{-j}} |Q|^{-s/n-1/2} |t_Q| |\chi_Q(x)| \right)^p \, dx \right\}^{1/q} < \infty.
$$

For $0 < p < \infty$, the space $\dot{F}_{p,q}^{s,\gamma}(\mathbb{R}^n)$ is defined as the collection of all sequences $t = \{ t_Q \}_{Q \in \mathcal{D}} \subset \mathbb{C}$, indexed by the dyadic cubes, such that

$$
\|t\|_{\dot{F}_{p,q}^{s,\gamma}} := \sup_{P \in \mathcal{D}} \frac{1}{|P|^\tau} \left\{ \int_P \left[ \sum_{Q \subset P} (|Q|^{-s/n-1/2} |t_Q| |\chi_Q(x)|)^q \right]^{p/q} \, dx \right\}^{1/p} < \infty.
$$

As before, we will use $\dot{A}_{p,q}^{s,\gamma}(\mathbb{R}^n)$ to denote either $\dot{B}_{p,q}^{s,\gamma}(\mathbb{R}^n)$ or $\dot{F}_{p,q}^{s,\gamma}(\mathbb{R}^n)$, excluding the case $p = \infty$ in the latter case.

Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$, $0 \leq \tau < \infty$ and $s^* := s - [s]$, where $[s]$ denotes the largest integer smaller than or equal to $s$. Set

$$
J := \left\{ \begin{array}{ll}
sp + n & \text{if } A_{p,q}^{s,\gamma}(\mathbb{R}^n) = \dot{B}_{p,q}^{s,\gamma}(\mathbb{R}^n) \\
spq + n & \text{if } A_{p,q}^{s,\gamma}(\mathbb{R}^n) = \dot{F}_{p,q}^{s,\gamma}(\mathbb{R}^n)
\end{array} \right.,
$$

where $s_p$ and $spq$ are as in (1.3). We say that $\{ m_Q \}_{Q \in \mathcal{D}}$, where $m_Q : \mathbb{R}^n \to \mathbb{C}$, is a family of smooth synthesis molecules for $A_{p,q}^{s,\gamma}(\mathbb{R}^n)$ if there exist $\delta$ and $M$ with $\max\{ s^*, (s + n\tau)^* \} <
\[ \delta \leq 1 \text{ and } J < M < \infty \text{ such that} \]
\[ \int_{\mathbb{R}^n} m_Q(x)x^\gamma \, dx = 0 \quad \text{if } |\gamma| \leq \max\{[J - n - s], -1\}, \]
\[ |m_Q(x)| \leq \frac{|Q|^{-1/2}}{(1 + l(Q)^{-1}|x - x_Q|)^{\max\{M, M - s\}}} \quad \forall x \in \mathbb{R}^n, \]
\[ |\partial^\gamma m_Q(x)| \leq \frac{|Q|^{-1/2 - |\gamma|/n}}{(1 + l(Q)^{-1}|x - x_Q|)^M} \quad \forall x \in \mathbb{R}^n \text{ and } |\gamma| \leq [s + n\tau], \]
\[ |\partial^\gamma m_Q(x) - \partial^\gamma m_Q(y)| \leq |Q|^{-1/2 - |\gamma|/n - \delta/n} |x - y|^{\delta} \]
\[ \times \sup_{|x| \leq |x|} \frac{1}{(1 + l(Q)^{-1}|x - z - x_Q|)^M} \quad \forall x, y \in \mathbb{R}^n \text{ and } |\gamma| = [s + n\tau]. \]

It easily follows that \( \{\varphi_{\nu,k}\}_{\nu \in \mathbb{Z}, k \in \mathbb{Z}^n} \) and \( \{\psi_{\nu,k}\}_{\nu \in \mathbb{Z}, k \in \mathbb{Z}^n} \) are families of smooth synthesis molecules for any \( \dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n) \) with parameters \( \delta = 1 \) and any \( M > J \).

Through analogous ideas on almost-diagonal operators used to prove [11, Theorem 3.5] it follows that if \( 0 < p, q \leq \infty, s \in \mathbb{R}, \max\{s^+, (s + n\tau)^+\} < \delta \leq 1, J < M < \infty, 0 \leq \tau < \min\{\frac{1}{p} + \frac{M - J}{2n}, \frac{1}{p} + \frac{1 - (J - s)^+}{n}\} \) if \( \max\{[J - n - s], -1\} \geq 0, 0 \leq \tau < \min\{\frac{1}{p} + \frac{M - J}{2n}, \frac{1}{p} + \frac{s + n - J}{n}\} \)
if \( \max\{[J - n - s], -1\} < 0, \) and \( m_Q \in \mathcal{E}(\mathbb{R}^n) \) is a family of synthesis molecules for \( \dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n) \) with parameters \( \delta \) and \( M, \) then
\[ (3.14) \quad \left\| \sum_{Q \in \mathcal{D}} t_Q m_Q \right\| \lesssim \|t\|_{\dot{A}_{p,q}^{s,\tau}} \quad \forall t = \{t_Q \}_{Q \in \mathcal{D}} \in \dot{A}_{p,q}^{s,\tau}, \]
where the implicit constant does not depend on the family of molecules ([29, Theorem 4.2]).

4. PROOF OF THEOREM 1.2 AND CLOSING REMARKS

Proof of Theorem 1.2. Let \( \varphi, \psi \in \mathcal{S}(\mathbb{R}^n) \) satisfy (3.10), (3.11) and (3.12). Since \( T_{\sigma_1} \) and \( T_{\sigma_2} \), as given by Theorem 1.1, are continuous from \( \mathcal{S}_0(\mathbb{R}^n) \times \mathcal{S}_0(\mathbb{R}^n) \) to \( \mathcal{S}(\mathbb{R}^n) \) and \( h = \sum_{\nu \in \mathbb{Z}, k \in \mathbb{Z}^n} \langle h, \varphi_{\nu,k}\rangle \psi_{\nu,k} \) for \( h \in \mathcal{S}_0(\mathbb{R}^n) \) with convergence in \( \mathcal{S}_0(\mathbb{R}^n) \) (see Section 3), we have
\[ T_{\sigma_1}(f, g) = \sum_{\nu \in \mathbb{Z}, k \in \mathbb{Z}^n} \langle f, \varphi_{\nu,k}\rangle T_{\sigma_1}(\psi_{\nu,k}, g) \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n), \]
\[ T_{\sigma_2}(f, g) = \sum_{\nu \in \mathbb{Z}, k \in \mathbb{Z}^n} \langle g, \varphi_{\nu,k}\rangle T_{\sigma_2}(f, \psi_{\nu,k}) \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n), \]
where the convergence is in \( \mathcal{S}(\mathbb{R}^n) \).

Theorem 1.1 implies that there are constants \( c_1 \) and \( c_2 \) such that if \( f, g \in \mathcal{S}_0(\mathbb{R}^n) \), then
\[ \left\{ c_1 2^{-mT_{\sigma_1}(\psi_{\nu,k}, g)} \right\}_{\nu \in \mathbb{Z}, k \in \mathbb{Z}^n} \quad \text{and} \quad \left\{ c_2 2^{-mT_{\sigma_2}(f, \psi_{\nu,k})} \right\}_{\nu \in \mathbb{Z}, k \in \mathbb{Z}^n} \]
are families of smooth synthesis molecules for any \( \dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n) \) if \( 0 < p, q \leq \infty, s > J - n \) and \( 0 \leq \tau < \infty \) (with \( \delta = 1 \) and any \( M > J \); note that the zero moment condition is void since
If, in addition, $0 \leq \tau < \frac{1}{p} + \frac{s+ n - 2}{n}$, we can apply (3.14) and (3.13) to get

$$
\|T_\sigma^1(f, g)\|_{\dot{A}_{p, q}^{s, \tau}} \lesssim \|2^{rm}(f, \varphi_{\nu, k})\|_{\dot{A}_{p, q}^{s, \tau}} \|g\|_{L^\infty} = \|\{f, \varphi_{\nu, k}\}\|_{\dot{A}_{p, q}^{s+ m, \tau}} \|g\|_{L^\infty},
$$

$$
\|T_\sigma^2(f, g)\|_{\dot{A}_{p, q}^{s, \tau}} \lesssim \|2^{rm}(g, \varphi_{\nu, k})\|_{\dot{A}_{p, q}^{s, \tau}} \|f\|_{L^\infty} = \|\{g, \varphi_{\nu, k}\}\|_{\dot{A}_{p, q}^{s+ m, \tau}} \|f\|_{L^\infty},
$$

from which the desired estimates follow.

Remark 4.1. Let $m \in \mathbb{R}$ and $\sigma \in \dot{B}^m_{1, 1}$. The estimates in Theorem 1.2 hold true in $\dot{A}_{p, q}^{s, \tau}$ for $0 < p, q \leq \infty$, $s \leq J - n$ and $0 \leq \tau < \frac{1}{p} + \frac{1 - (J - s)^*}{n}$ if the following cancellation conditions are satisfied:

$$
T_\sigma^1(x^\gamma, g) = T_\sigma^1(f, x^\gamma) = 0 \quad \forall f, g \in S_0(\mathbb{R}^n), \quad |\gamma| \leq |J - n - s|.
$$

We recall that if $T$ is a bilinear operator continuous from $S_0(\mathbb{R}^n) \times S_0(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$, $T^{*1}$ and $T^{*2}$ denote the adjoint operators of $T$ defined from $S'(\mathbb{R}^n) \times S'_0(\mathbb{R}^n)$ to $S'_0(\mathbb{R}^n)$ and from $S'_0(\mathbb{R}^n) \times S'(\mathbb{R}^n)$ to $S'_0(\mathbb{R}^n)$, respectively, as $\langle h, T(f, g) \rangle = \langle T^{*1}(h, g), f \rangle = \langle T^{*2}(f, h), g \rangle$.

The proof of the estimates in this case is the same as above, with the only thing left to check being the zero moment conditions for $T_\sigma^1(\psi_{\nu, k}, g)$ and $T_\sigma^2(f, \psi_{\nu, k})$ (note that the range assumed for $\tau$ comes from the assumptions for the validity of (3.14)). We have, for $|\gamma| \leq |J - n - s|$,

$$
\int_{\mathbb{R}^n} x^\gamma T_\sigma^1(\psi_{\nu, k}, g) \, dx = \langle x^\gamma, T_\sigma^1(\psi_{\nu, k}, g) \rangle = \langle T_\sigma^1(x^\gamma, g), \psi_{\nu, k} \rangle = 0 \quad \forall g \in S_0(\mathbb{R}^n),
$$

and similarly for $T_\sigma^2(f, \psi_{\nu, k})$.

Remark 4.2. Let $1 \leq r \leq \infty$ and $m$, $\sigma$, $p$, $q$, $s$ and $\tau$ be as in the hypothesis of Theorem 1.2 or Remark 4.1. By the same reasoning as in the proof of Theorem 1.2 and Remark 4.1, we also obtain

$$
\|T_\sigma(f, g)\|_{\dot{A}_{p, q}^{s, \tau}} \lesssim \|f\|_{\dot{A}_{p, q}^{s+ m + \frac{2}{p}, \tau}} \|g\|_{L^{\infty}} + \|g\|_{\dot{A}_{p, q}^{s+ m + \frac{2}{p}, \tau}} \|f\|_{L^r}.
$$

Remark 4.3. The implicit constants in the inequalities of Theorem 1.1 and Theorem 1.2 depend linearly on $\|\sigma\|_{K, L}$ for some $K, L \in \mathbb{N}$, where

$$
\|\sigma\|_{K, L} := \sup_{|\gamma| \leq K, |\alpha + \beta| \leq L} \|\sigma\|_{\gamma, \alpha, \beta}.
$$

From the proofs, it follows that the implicit constants in the inequalities of Theorem 1.1 are multiples of $\|\sigma\|_{|\gamma|, 2N}$, with $N \in \mathbb{N}$, $N > M + n$ and where $\gamma$ and $M$ are as in the statement of the theorem. In turn, this implies that the implicit constants in Theorem 1.2 can be taken to be multiples of $\|\sigma\|_{|s + n\tau|, 1, 2N}$ with $N > \max\{J + n, 2(s + n) - J + n\}$. The latter is also true for the inequalities from Remark 4.1 with $N > J + n + 2(1 - (J - s)^*)$.

References


