Chain Rule I:
Let \( x = x(t) \) and \( y = y(t) \) be differentiable at \( t \), and let \( z = f(x,y) \) be differentiable at \( (x(t), y(t)) \). Then \( z = f(x(t), y(t)) \) is differentiable at \( t \) and
\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]

Chain Rule II:
Let \( x = x(s,t) \) and \( y = y(s,t) \) be differentiable at \( (s,t) \), and let \( z = f(x,y) \) be differentiable at \( (x(s,t), y(s,t)) \). Then \( z = f(x(s,t), y(s,t)) \) is differentiable at \( (s,t) \) and has partial derivatives given by
\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}
\]

Second Derivative Test:
Assume that \( f(x,y) \) is twice continuously differentiable in an open set containing \( \vec{p}_0 := (x_0, y_0) \), and suppose that \( \nabla f(\vec{p}_0) = (0,0) \). Set
\[
D = D(\vec{p}_0) := f_{xx}(\vec{p}_0)f_{yy}(\vec{p}_0) - f_{xy}(\vec{p}_0)^2.
\]
Then
(i) If \( D > 0 \) and \( f_{xx}(\vec{p}_0) < 0 \), then \( \vec{p}_0 \) is a local maximum.
(ii) If \( D > 0 \) and \( f_{xx}(\vec{p}_0) > 0 \), then \( \vec{p}_0 \) is a local minimum.
(iii) If \( D < 0 \), then \( \vec{p}_0 \) is a saddle point.
(iv) If \( D = 0 \), then the test is inconclusive.

Lagrange Multipliers I:
The minimum and maximum values of \( f(\vec{p}_0) \) subject to the constraint \( g(\vec{p}_0) \equiv 0 \), satisfy \( \nabla f(\vec{p}_0) = \lambda \nabla g(\vec{p}_0) \) for some scalar value \( \lambda \). (\( \lambda \) is called the Lagrange multiplier.) The possible extreme values can then be found among the solutions of the system:
\[
\nabla f(\vec{p}_0) = \lambda \nabla g(\vec{p}_0) \quad \text{and} \quad g(\vec{p}_0) \equiv 0.
\]

Lagrange Multipliers II:
The minimum and maximum values of \( f(\vec{p}_0) \) subject to the two constraints \( g(\vec{p}_0) \equiv 0 \) and \( h(\vec{p}_0) \equiv 0 \) satisfy \( \nabla f(\vec{p}_0) = \lambda \nabla g(\vec{p}_0) + \mu \nabla h(\vec{p}_0) \) for some scalar values \( \lambda \) and \( \mu \). The possible extreme values can then be found among the solutions of the system:
\[
\nabla f(\vec{p}_0) = \lambda \nabla g(\vec{p}_0) + \mu \nabla h(\vec{p}_0), \quad g(\vec{p}_0) \equiv 0, \quad \text{and} \quad h(\vec{p}_0) \equiv 0.
\]