Functions, Graphs, and Graphing: Tasks, Learning, and Teaching

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This review of the introductory instructional substance of functions and graphs analyzes research on the interpretation and construction tasks associated with functions and some of their representations: algebraic, tabular, and graphical. The review also analyzes the nature of learning in terms of intuitions and misconceptions, and the plausible approaches to teaching through sequences, explanations, and examples. The topic is significant because of (a) the increased recognition of the organizing power of the concept of functions from middle school mathematics through more advanced topics in high school and college, and (b) the symbolic connections that represent potentials for increased understanding between graphical and algebraic worlds. This is a review of a specific part of the mathematics subject matter and how it is learned and may be taught; this specificity reflects the issues raised by recent theoretical research concerning how specific context and content contribute to learning and meaning.

Introduction

This paper is a review of research and theory related to teaching and learning in a particular subject, mathematics; in a particular domain, functions, graphs, and graphing; at a particular age level, 9–14. Major reviews in education are rarely so embedded. Indeed, since 1970, there has been no subject topic-specific review in Review of Educational Research. Recent theoretical thinking and research in

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cognitive psychology, cognitive anthropology, educational psychology, and philosophy, however, suggest that there are aspects of learning and teaching specific content that are unique to or more salient to the particular topic than to the field of teaching and learning as a whole. (See Collins, Brown, & Newman, 1989; Greeno, 1988b; Lave, 1988; Shulman, 1986, 1987.) Clearly, if each specific situation is totally unique, then one cannot hope to build a body of accumulated knowledge, let alone science. There is a tension between the specific and the general (Schwab, 1964). It appears at this point, however, that the cumulated study of the specific may enrich our conceptualization of the general. It is from this perspective that we offer this review.

We focus on functions and graphs for several reasons. First, much of the research on mathematics learning and teaching has focused on the very earliest levels of mathematics content. Functions and graphs, on the other hand, is a topic that generally does not appear until the upper elementary grades or later. Second, functions and graphs represent one of the earliest points in mathematics at which a student uses one symbolic system to expand and understand another (e.g., algebraic functions and their graphs, data patterns and their graphs, etc.). Third, graphing can be seen as one of the critical moments in early mathematics. By moments we mean sites within a discipline when the opportunity for powerful learning—different in kind from other episodes—may take place. Other moments are acquiring notions of transformation inherent in regrouping, expanding the number system from counting numbers to rationals, moving from additive to multiplicative structures, and so forth. These moments have two interesting features: They are often unmarked in the “normal” course of teaching; on the other hand, they are fundamental to other more sophisticated parts of mathematics.

The assumption of research that has focused on the earliest grades and concepts of mathematics seems, in some cases, to have been that we can reason upward from the findings of learning in elementary mathematics. The research on early mathematics has emphasized the significance and power of embedding abstract mathematical concepts in the real, intuitive, experiential knowledge of young children. For example, children know about part/whole from sharing and dividing clay, juice, or cookies. Because part/whole notions are fundamental to concepts of addition and subtraction (Resnick & Omanson, 1987; Riley, Greeno, & Heller, 1983), building on these early intuitive ideas is seen to be helpful in the acquisition of the abstract, more formal arithmetic. This is one of several ideas that supports Cognitively Guided Instruction (CGI) (Fennema, Carpenter, & Peterson, in press). Mathematics, however, also consists of abstractions and formalisms, which at some point become the basis from which one reasons rather than that to which one reasons. Thus, students readily and sometimes inappropriately apply whole number principles to fractions and decimals (not just procedures but the underlying meanings as well) (Resnick, 1989). If students could not reason from prior mathematical knowledge, mathematics would be immeasurably more difficult. The fact that they sometimes do it inappropriately makes teaching more complex.

Although functional relationships have been recognized for some time as important constructs in the development of abstract mathematical knowledge (Piaget, Grize, Szeminska, & Bang, 1968/1977), functions and graphs have not been the object of much intellectual scrutiny by the educational community until recently (in the way, for example, addition and subtraction, fractions, algebra word prob-
functions, or even negative numbers have been). There is now, however, a growing body of research and questions surrounding the topic. Individuals from a variety of disciplines, including mathematics, mathematics education, and cognitive psychology, are conducting this work. We believe it is an opportune time to pull together the existing body of work on functions and graphs because, although such an integration must span several disciplines that have not been in contact previously, the number of studies is still small enough to make such an integration feasible and informative for future work in this area.

One of the features of the domain that has drawn us to the study of functions and graphs is that algebraic and graphical representations are two very different symbol systems that articulate in such a way as to jointly construct and define the mathematical concept of function. Neither functions nor graphs can be treated as isolated concepts. They are communicative systems, on the one hand, and a construction and organization of mathematical ideas on the other. Although much of the prior mathematical work in the student’s life may have dealt with concrete representations as the basis for learning more abstract concepts, functions and graphs is a topic in which two symbolic systems are used to illuminate each other. This feature places demands on the learner in terms of new ideas, notational uniqueness, and symbolic correspondences. It does not, however, preclude a more intuitive or experiential foundation. (Indeed, Piaget et al., 1968/1977, suggest the presence of intuitive understanding of function.) Rather, it means that in this topic we have a case in which two symbol systems both contribute to and confound the development of understanding. It thus can serve as an interesting bridge between reasoning from the concrete to the abstract and reasoning among abstractions.

The bridge between functions and graphs is also interesting because the intellectual landscape, so to speak, looks different from each side of the bridge—if graphs are used to explicate functions, the sense of function (and graph) is quite different from what is presented the other way around. Indeed, part of the problem with graphing as a scientific tool resides in this issue of directionality of thinking about graphs and functions. The mathematical presentation is usually from an algebraic function rule to ordered pairs to a graph, or from a data table of ordered pairs to a graph. The scientific presentation, on the other hand, most often proceeds from observation, to data array, to ordered pairs of data, to selection of axis labels, to scale construction, to graph and (maybe) to function. Often, students who can solve graphing or function problems in mathematics seem to be unable to access their knowledge in science. It is only recently with the insights of cognitive science that we are beginning to learn why the truism holds: Just because learners know something in one way does not mean that they can make immediate use of it from a different perspective or in a different situation (Greeno, 1988a, 1988b). Because of its presence and its distinctive character in both mathematics and science, functions and graphs provides an excellent topic to examine in this respect. Finally, functions and graphs are interesting in an instructional sense because they tend to focus on relation as well as entity, and because they are a magnificent tool for examining patterns. It is in this role that the powerful use of the computer is especially salient.

The topic of functions and graphs is nestled in the elementary curriculum of mathematics.1 It is a marginal, extra topic in most commonly used commercial textbooks for most of the elementary years (through sixth grade). Most texts
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approach graphing as information display, usually in the form of bar graphs, pictographs, circle graphs, and line graphs (Stein & Baxter, 1989). In addition, notational work related to the Cartesian coordinate system often is done for a few pages each year. Despite calls for the introduction and development of functions in the elementary years (e.g., National Council of Teachers of Mathematics, 1989), the vast majority of commercial textbooks do not teach graphing in relation to functions. (One exception is Open Court’s Real Math [Willoughby, Bereiter, Hilton, & Rubinstein, 1981], which includes a function strand beginning in first grade. By third grade, Real Math introduces graphing in relation to functions and continues this study through the rest of elementary school.) After this generally limited treatment in the elementary years, functions and graphs blossom in the latter part of algebra I and early algebra II and then disappear until calculus. On the other hand, graphs without explicit functions are a stable presence in science and social studies texts from middle school through high school.

The literature on functions and graphs can be considered from several perspectives. One perspective is to consider an analysis of the tasks and their presentation. Central ways of looking at the task of graphing and functions include issues of focus, notational convention, and ambiguity surrounding unit and scale. Another perspective is to consider the learner and the development of understanding about graphs and functionality. Learners have intuitions, misconceptions, and other difficulties surrounding graphs and functions. A third perspective is to consider how functions, graphs, and graphing are taught in classroom settings and in computer environments. Here the consideration is with the role of teachers’ subject matter knowledge, the tactical approach or entry to instruction, and the construction of explanations.

Task

We discuss the issue of task in functions, graphs, and graphing from four overlapping constructs: the action of the student or learner, the situation, the variables and their nature, and the focus. By action we refer to first whether the task is interpretation (e.g., reading, gaining meaning) or construction (e.g., plotting a graph from a data set, determining an equation from a graph, or generating an example of a function). We then analyze four typical tasks that involve interpretation or construction: (a) prediction tasks, which rely mostly on construction and address the issue of pattern; (b) classification tasks, which require interpretation and address the definition and special properties of functions; (c) translation tasks, which can be either interpretation or construction and address the issue of representations; and (d) scaling tasks, which also can be either interpretation or construction and involve decisions regarding scale and unit that are characteristic in particular to the domain of graphing. Scale refers to the assignment of values to intervals between the lines on the Cartesian system. In two-dimensional graphs, there are two decisions, one for the x-axis and one for the y-axis. Unit is a notion connected to scale. It refers to the thing that is counted or measured (e.g., one wiggit). Unit is an issue in situated contexts such as science.

Situation refers to the setting of the graph or function and can be more or less contextualized or abstract. The nature of the situation influences the interpretation and the plausibility of results and the type of variables used. Situation refers to
both the context of the graph and the setting in which the graph is being used—a science lab, mathematics class, or social studies.

Variables are the objects of functions and graphs; they are the data, both concrete and abstract. In two-dimensional graphs the variables may be in the same form or in different forms. By form we are referring to the property of the unit, whether it is categorical, ordinal, interval, or ratio. Another feature of a variable has to do with whether it is discrete or continuous. A ratio form is associated with continuous variables, whereas the other three forms are of a discrete nature.

Focus is tied to both situation and action. Focus refers to the location of the attention within a specific task. The focus can be primarily internal to the coordinate system, that is, on the graph and its components; or it can be on the coordinate system, that is, the axes and their labels and scales.

Student Learning

We turn now to an overview of key issues related to student learning in the area of functions and graphs. Although the focus of this section is on the individual as a learner, notions of task are embedded implicitly in the discussion—the student learns the domain through his or her engagement in tasks. The discussion on student learning is organized into two main sections: intuitions and misconceptions.

Intuitions are features of a student’s knowledge that arise most commonly from everyday experience; in general, they are seen to exist prior to specific formal instruction. The most recent thinking in mathematics education views intuitions as positive and as occasions around which to build instruction and learning (Fischbein, 1973, 1978; Resnick, 1989). The topic of functions (in its more advanced form) is extremely complex. This is due to several factors, including the following: (a) It is often associated with other complex mathematical concepts (i.e., variable, growth, limit, extremum, pictorial meaning); (b) it is integrative in nature, pulling together various subconcepts and fields of mathematics; and (c) it appears in many different representations (Dreyfus & Eisenberg, 1982). Because of the complexity of the concept of function, tracing the roots of mature functional thinking to children’s intuitions involves searching for multiple sources. Indeed, it is not unusual to find an array of seemingly unrelated intuitions discussed in the literature. This is natural, however, because by nature intuitions are only partly realized and poorly coordinated. The task for instruction and for learning is to expand and pull together these seemingly disparate threads into a unified, mature notion of function.

Misconceptions are features of a student’s knowledge about a specific piece of mathematics knowledge that may or may not have been instructed. A misconception may develop as a result of overgeneralizing an essentially correct conception, or may be due to interference from everyday knowledge. To qualify, a misconception must have a reasonably well-formulated system of ideas, not simply a justification for an error. So although misconception does not need to be an entire theory, it should be repeatable and/or explicit rather than random and tacit.

Some misconceptions can be traced logically to intuitions. For example, students’ tendency to interpret graphs iconically may be related to their intuitions regarding picture reading. Other misconceptions can be interpreted as the result of incomplete formal learning. An example of this type of misconception would be students’
tendency to recognize only one-to-one correspondences as functions (omitting many-to-one functions to which they may not have been exposed).

Teaching

Instruction in the domain of graphs and functions is informed necessarily by issues related to task and to student learning. How one teaches functions, graphs, and graphing is a consequence of the structure and idiosyncrasies of the domain as well as the teacher’s knowledge of how student understanding develops in the domain. Nevertheless, informed pedagogy also draws on knowledge that extends beyond knowledge of the task and knowledge of how students learn. It must be sensitive to decisions about entry, sequencing, explanations, examples, and representations.

There is no proven optimal entry to functions and graphs. It can be entered through graphing and grid knowledge obtained by students in social studies or science classes. It also can be entered through a knowledge development of rules of correspondence, such as in Open Court’s Real Math series. It can be entered through a notational exercise of point location and transformation of shapes, such as in many texts—essentially a playful notion of graphing that can be built on later. Obviously, the selection of entry dictates to some extent the subsequent path of instruction and learning, or sequencing. Again, dependent on sequencing, explanations are built that take advantage of or build on prior knowledge. Explanations consist of the orchestrations of demonstrations, analogical representations, and examples. Basic notions of how explanations can be best designed to capitalize on what students know and already understand have been discussed in other parts of mathematics but not for this topic (Leinhardt, 1988, in press). A primary feature of explanations is the use of well-constructed examples, examples that make the point but limit the generalization, examples that are balanced by non- or counter-cases.

The goal of instruction is to develop student conceptions. Conceptions are features of a student’s knowledge about a specific and usually instructed piece of mathematics. They are meaningful ideas that students develop that can serve as powerful workhorses in the student’s continued efforts to reach deeper, more integrative levels of understanding. Conceptions are embedded in the domain in the sense that they consist of content that is specific to understanding in a particular domain. By nature, conceptions are in transition, or in the process of being fleshed out to their fullest realization or capacity. (This is why they can at times appear to be fragile, situation-bound, or misapplied.) Nevertheless, they are well enough structured to do work for the student and explicit enough to be the object of communication with others. In their ideal form, conceptions are highly interactive with supportive intuitions. But they also can be actively split from intuitions when they cannot easily fit into the intuitive structure; hence it is easier for the student to keep two separate islands of knowledge.

The basic structure of this review can be visualized as a staircase. The first step consists of our analysis of the nature of the task. What are the actions and meanings associated with beginning work in the domain of functions and graphs? The second step builds on all the topics and issues raised in the discussion of task but adds to them what is known about the learners who face the problem of acquiring the knowledge mapped out previously. In addition to the nature of the domain, what
do educators need to know and understand about learners to design a successful instructional program? Finally, the third step builds on the previous two by investigating what additional information is available concerning teaching per se. Here the literature is incomplete, but salient work from other areas of teaching mathematics and inferences from research on teaching and learning in functions and graphs can be used.

A note on technology. More than perhaps any other early mathematics topic, technology dramatically affects the teaching and learning of functions and graphs. Careful consideration of this area would require another review of equal length, which is beyond the scope of this paper. What we do urge is a research concern for the unique problems that can develop because of technology, as well as an exploration of its advantages.

The availability of graphical technologies affects issues of concern to teachers and curriculum developers as well. With these technologies students have access much earlier in the curriculum to more complex notions related to graphs. For example, lack of familiarity or competence with algebraic techniques does not prevent a student from exploring complicated computer-generated graphs and from being engaged in problem-solving activities that follow. This may point to a need to alter the instructional sequence. Properties of a graph, such as continuity, may need to be addressed earlier and in a manner different from the traditional one. There is also an issue of authority and source of information. In computer-based instruction, the authority may be in the hands of the computer in subtle ways that do not support learning. The graph that the machine produces is unquestionable. A teacher should be aware of the “magic” effect this may have on students. Students may develop too strong a reliance on the machine. For example, they may not understand the underlying patterns and principles that drive the production of the graphs. A breakthrough in teaching and designing learning opportunities that graphical technologies make possible is the use of graphs to describe, in real time, data from scientific experiments. This may help math teachers facilitate experiences that interconnect with science.

The issue of multiple representations may become more salient in the context of these technologies. Working simultaneously with at least two linked representations is more manageable with these media. Most graphical technologies provide a graphical representation of a function and can display simultaneously at least two representations, such as an equation and a graph, or an ordered pair of numbers and a point on a graph. Some also give a table of values, which are the data that the graph builds on.

Even though the task seems to change, there are salient features of the domain that should be considered in teaching, regardless of the media. These features include issues of representation, multiple representations, the extent to which a visual representation (graph) is symbolic, the conventions that go with each symbolic system, and the freedom of choice (scale, focus). Little research has been done on the teaching of graphing and functions through technologies. Most of what is reported are criteria for development of software and theoretical considerations.

Task

We turn in this section to a deeper analysis of the tasks that appear in the literature on functions and graphs. This section consists of four main parts by
which the tasks are analyzed: action, situation, variable, and focus. The complexity of this analysis mirrors the complexity of the domain of graphs and functions. As mentioned in the introduction, this domain is not as well defined and structured with respect to the task as is the relatively more explored territory of addition and subtraction, or even that of multiplicative structures.

**Action**

Most of the actions that relate to graphs and functions tasks can be classified into two main categories: interpretation and construction. (These are not exhaustive or mutually exclusive categories.) Some type of interpretation and/or construction is at the heart of most of the commonly used tasks. The majority of the extant studies look mainly at interpretation tasks (Barr, 1980; Bell & Janvier, 1981; Bestgen, 1980; Clement, 1989; Dreyfus & Eisenberg, 1982; Janvier, 1981a; Karplus, 1979; Kerslake, 1981; Krabbendam, 1982; McKenzie & Padilla, 1986; Monk, 1987; Preece, 1983a; Vinner, 1983). Only a few focus on construction tasks (Dreyfus & Eisenberg, 1983; Krabbendam, 1982; Wavering, 1985).

Interpretation and construction actions can be either fairly straightforward or quite complex, depending on several factors that will be discussed. In this section we analyze interpretation and construction actions along two dimensions: a “local-to-global” dimension and a “quantitative-to-qualitative” dimension. Then we discuss in detail four typical tasks with respect to the interpretation and construction that they require. These tasks are prediction, classification, translation, and scaling.

**Interpretation.** By interpretation we refer to the action by which a student makes sense or gains meaning from a graph (or a portion of a graph), a functional equation, or a situation. Interpretation can be global and general or it can be local and specific. Thus, a student may be trying to decide issues of pattern (e.g., What happens to $x$ as $y$ increases?), continuation (e.g., interpolation or extrapolation of a graph), or rate (e.g., How do the bacteria change after every 5 hours at a fixed temperature?); or determining when specific events or conditions are met (e.g., What is the minimum? At what point does the car round a curve? Who is the tallest child in the class?). Interpretation actions are subject to both over- and undergeneralizations and to confusion or confounding with pictorial events of a similar nature.

When interpreting a graph, the kind of interpretation required of the student depends largely on what the graph represents. The graphs that are used most commonly in studies represent either a situation or an abstract functional relationship (usually expressed by an equation, and sometimes represented by a table of ordered pairs), or they are considered entities in their own right. Depending on what the graph stands for, the meaning gained by interpretation can either reside within the symbolic space of the graph, or it can shift to a different space (the situation space or the algebraic space). Interpretation that requires moving from one representation form to another is, according to Janvier (1987d) and Kaput (1987c), a type of translation.

Figure 1 describes the movement among spaces that can occur in an interpretation action. (Any interpretation task involves to a certain extent interpretation within the space that is the focus of the task, be it a graph, a problem situation, or an algebraic rule. Some tasks involve interpretations between two spaces as well.
However, in the literature there are rare occasions that require direct interpretation between a specific situation and an algebraic rule.

Thus, looking at a given graph and determining where its $y$-intercept is or deciding what points have negative coordinates requires interpretation within the coordinate system in which the graph is embedded. On the other hand, determining from a linear graph that the value of $m$ (the slope) in the corresponding equation ($y = mx + b$) is positive involves both interpreting the graph itself and shifting (often back and forth) from the graph to the equation (Schoenfeld, Smith, & Arcavi, in press). Another type of task in which the interpretation also occurs outside the graph is when the graph describes a specific situation. In this case interpreting it involves shifting from the graphical representation to the situation itself.

Interpretation tasks tend to involve graphs that represent situations (Janvier, 1980, 1981a, 1981b; Preece, 1983a). Given a specific graph representing a situation, there are a variety of questions that can be asked about it, all involving interpretation. Depending on the questions asked, the interpretation can be a local process (e.g., one regarding point-by-point attention) or a more global one (e.g., trend detection). The following example in Figure 2 (taken from Bell & Janvier, 1981, p. 37) illustrates how the interpretation process can move from local, that is, point reading (questions 1 and 2), to more global, that is, interval reading (questions 3 through 6), and finally, to global in its fullest sense, that is, whole graph reading (question 7).

There are many global features of a graph that can be interpreted. These include general shape of the graph, intervals of increase or decrease, and intervals of extreme increase or decrease. It should be noted that one can attend to global features of a graph whether the graph represents a specific situation, as in Figure 2, or whether it represents an abstract functional relationship. Even though understanding global features of a graph is valuable both for more advanced mathematics (especially in calculus) and for a full understanding of the situations represented by

![FIGURE 1. Movement among spaces that can occur in an interpretation action](image-url)
1. The average weight of boys at age 9 is ..................
2. The average weight of girls at age 17 is ..................
3. From what age do the boys on average weigh
   more than 55 kilograms? ..............................
4. From what age do the girls on average weigh
   more than 20 kilograms? ..............................
5. When (at what ages) do the girls weigh
   more than the boys? .................................
6. By how many kilograms does the average weight of girls increase
   between age 3 and age 8? ............................
7. At what age do the girls put on
   weight most rapidly? ...............................
graphs, treatment of these features generally is overlooked in the earlier grades of the curriculum. Several authors (Bell & Janvier, 1981; Janvier, 1980; Kerslake, 1981; Monk, 1987; Preece, 1983a) have argued that there is disproportionate emphasis in the curriculum on tasks involving local interpretations. Overemphasizing pointwise interpretations may result in a conception of a graph as a collection of isolated points rather than as an object or a conceptual entity (Schoenfeld et al., in press; Stein, Baxter, & Leinhardt, in press; Yerushalmy, 1988).

Another dimension along which interpretation tasks can be analyzed is their progression from a quantitative to qualitative interpretation. Figure 3 is an example taken from McKenzie and Padilla (1986, p. 575) that illustrates what qualitative interpretation tasks involve.

A qualitative interpretation of a graph in its fullest sense requires looking at the entire graph (or part of it) and gaining meaning about the relationship between the two variables and, in particular, their pattern of covariation. The tasks in Figure 3 draw students' attention to the general trend of the graph (e.g., "as the pot size increases, the plant size decreases") rather than to exact quantities (e.g., What was the increase of the size of plant when the size of pot increased by a certain number of units?). Qualitative interpretation often is associated with global features. Although global features can be interpreted either quantitatively (Figure 2) or qualitatively (Figure 3), it is less common to interpret local features qualitatively, except for dramatic changes of shape, rate, or direction.

Qualitative interpretation of graphs is another underrepresented area in the mathematics curriculum. A number of studies that investigate students' qualitative interpretation of graphs emerge from a science orientation (Brasell & Rowe, 1989; McDermott, Rosenquist, & vanZee, 1987; McKenzie & Padilla, 1986; Mokros &

Dick plans to study the effect of growing sunflowers in different size pots. The graphs below show four possible outcomes of his experiment.

Which graph is best described by each of the following statements.

16. As the pot size increases, the plant height decreases.

17. As the pot size increases the plant height increases up to a certain pot size. With larger pots, plant height remains the same.

FIGURE 3. A qualitative interpretation task

Tinker, 1986) or from a point of view that mathematics could provide tools to help explain physical phenomena and that students should be able to use mathematics for this purpose (Janvier, 1981b; Preece, 1983a). Other tasks that focus in particular on qualitative interpretation of graphs are those that emerge from computer environments (Demana & Waits, 1988; Goldenberg, 1987; Yerushalmy, 1988). These studies build on the computer's capability to free students from much of the computational and technical work, thus enabling them to focus on more qualitative and global features of the graph.

To summarize, interpretation actions can vary depending on where most of the action takes place (entirely within the graph or also outside it, Figure 1). They also can vary depending on the features of the graph that need to be attended to (i.e., local-to-global, quantitative-to-qualitative). We now turn to the action of construction.

Construction. By construction we refer to the act of generating something new. Construction refers to building a graph or plotting points from data (or from a function rule or a table) or to building an algebraic function for a graph. In its fullest sense, construction involves going from raw data (or abstract function) through the process of selection and labeling of axes, selection of scale, identification of unit, and plotting. In the past, construction has been a rather tedious, point-by-point undertaking. With the advent of technology, however, parts of these actions can be performed by a computer in a more global fashion, and need not be the task of the learner. Indeed, the task domain of graphing and functions lends itself well to the new interactive graphical technologies. These technologies allow the student to examine many more graphs more quickly with an extremely high degree of accuracy. It is important to note, however, that releasing the student from constructing a graph from scratch with paper and pencil alters the task domain enormously. For example, the mathematical approach to solving problems changes. As stated by Demana and Waits (1988), “While the traditional approach uses careful numerical and algebraic investigation to produce the graph of a function, computer-generated graphs of a given function can now be used to determine its important numerical and algebraic properties” (p. 177). In addition, the direction of the activity moves toward interpretation of a global behavior of a graph and toward generalization from a particular visible part of a graph to other parts of it, or from a number of graphs that share a common feature to a wider class of functions of which they are instances. Even more so than in the traditional approach, this raises the question, To what extent can what is actually seen in a graph be relied on? The part of the graph displayed on the screen may be misleading in predicting the behavior of the whole graph.


Construction is quite different from interpretation. Whereas interpretation relies on and requires reaction to a given piece of data (e.g., a graph, an equation, or a data set), construction requires generating new parts that are not given. Similar to
interpretation, construction can be fairly simple or quite difficult. For example, plotting points from a table of ordered pairs, once the axes and scales are set up, is quite straightforward. Constructing an equation that represents a given graph can be rather difficult, however, mainly because it is not obvious where one should focus and what implicit givens would be useful for determining the equation (Markovits et al., 1986; Stein & Leinhardt, 1989; Yerushalmy, 1988). Indeed, constructing an equation from a graph is not as common in the literature (Her-scovics, 1982, is one of the few that directly addresses such tasks) as is the reverse—constructing a graph from an equation, which often is done pointwise via tables of ordered pairs. The difficulty of the task of constructing an equation from a graph is further evidenced by the difficulties that students do, in fact, encounter (see “representations” in student learning section).

Construction, like interpretation, can be either local [e.g., plot a number of points (Kerslake, 1981; Wavering, 1985)], or global [e.g., complete a graph given a number of its points (Karplus, 1979; Markovits et al., 1986)]. Construction also can be quantitative [e.g., determine the coefficients m and b in the equation \( y = mx + b \) given two points that satisfy this equation (Stein & Leinhardt, 1989)] or qualitative [e.g., sketch a graph representing a situation (Krabendam, 1982; Preece, 1983b; Swan, 1982)]. As with interpretation, attention to qualitative features more often is connected to global as opposed to local features.

To conclude, constructions and interpretations share a number of characteristics. In both, the action can take place in any one of the spaces (the graphical, the algebraic, or the situation) or move from one space to another. In addition, both construction and interpretation can vary with respect to the features that are being attended to (local-global, quantitative-qualitative). In terms of their relationship to each other, it can be noted that whereas interpretation does not require any construction, construction often builds on some kind of interpretation.

Having explored what interpretation and construction mean, we now turn to four kinds of tasks that involve interpretation and/or construction: prediction, classification, translation, and scaling. These tasks are typical of the domain of graphing and functions and are representative of the wide range of tasks addressed in the literature.

**Prediction.** By prediction we refer to the action of conjecturing from a given part of a graph where other points (not explicitly given or plotted) of the graph should be located or how other parts of the graph should look. Prediction also can occur when conjecturing a rule given a number of its instances [e.g., “guess my rule” (Davis, 1982; Swan, 1982; Willoughby et al., 1981)]. (Studies that incorporate prediction tasks include Bell & Janvier, 1981; Dreyfus & Eisenberg, 1983; Greenero, 1988a, 1988b; Karplus, 1979; Markovits et al., 1983, 1986; Stein & Leinhardt, 1989.) At the heart of most prediction tasks is an action of construction, which can be done either physically or mentally.

Close examination of prediction tasks reveals that not all such tasks rely on the same set of skills. Some prediction tasks rely primarily on estimation and to some extent on measurement skills, whereas others depend on pattern detection. In Bell and Janvier’s task (Figure 2), for example, the questions that ask for the average weight given the age (questions 1 and 2) require interpolation. In this case, the pattern of the lines does not provide enough quantitative information to determine exactly the average weight of the boys at age 9. In addition, the coordinates of the
target points cannot be read off the graph. Therefore, the interpolation is based on estimation that may also involve some measurement activity.

In other cases, prediction tasks involve pattern detection, either in a contextualized science situation (Karplus, 1979) or in an abstract situation (Stein & Leinhardt, 1989). Here the task is to determine the location of and to plot points that do not appear on the original graphs. For example, in Karplus’s (1979) Bacteria Puzzle, students were given data about the final growth of bacteria cultures (in terms of their diameter) at different temperatures. The data consisted of five ordered pairs: 

- $(0^\circ C, 2\text{mm})$
- $(10^\circ C, 6\text{mm})$
- $(30^\circ C, 34\text{mm})$
- $(35^\circ C, 29\text{mm})$
- $(40^\circ C, 4\text{mm})$

The students were asked to predict the final culture diameters at different temperatures: $20^\circ C$, $25^\circ C$, and $33^\circ C$. This task could be approached by numerical interpolation or through graphical interpolations (space for calculation and labeled graph grid were provided). In either case, this task required that students detect a pattern (curvilinear) based on the given points and then construct points that followed that same pattern. This act of construction depends on the student’s interpretation of the situation (i.e., the behavior of the phenomena) as well as on the student’s belief of what a graph representing such a situation should look like.

In the Stein and Leinhardt (1989) study, students also were asked to plot a point (given only its $x$-coordinate) that would follow the same pattern shared by a number of other points, in this case, all lying on a straight line. With the absence of an equation of the line or a known procedure by which the students could determine its equation, this task involved detecting the pattern of the line (e.g., when $x$-coordinate increases by 2 units, the $y$-coordinate increases by 4 units) and using this pattern to determine the location of the unknown point. Note that even interpolation tasks differ from each other with respect to the kinds of demands they present to students (from estimation and measurement to pattern detection).

A characteristic of many prediction tasks is that they cannot be tested (or checked). In addition, they often have no one right answer. Some predictions may be better than others, but usually there is not enough information (at least not explicitly) that determines one exact solution. Prediction tasks vary with respect to the extent of diversity of acceptable answers. In Stein and Leinhardt’s (1989) example, as in Bell and Janvier’s (1981) example (Figure 2), the boundaries of the answer are given implicitly by the surrounding points on the line. In Karplus’s (1979) example the boundaries are much looser. In the following example (see Figure 4), however, theoretically any answer is acceptable. In this task, Dreyfus and Eisenberg (1983, p. 127) asked students to continue the given graphs. The authors used this task to identify the properties or patterns of graphs that students were most attracted to (e.g., linearity, smoothness, periodicity).

Once again, the way a student constructs the rest of the graph depends heavily on how he or she thinks a graph should look (see “what is and what is not a function” in the student learning section).

Prediction tasks also can be used for functions that are not part of the graphical world (Greeno, 1988a). While operating on a special kind of function machine that behaves as a linear function, students were asked to predict what will happen when they turned the handles, how far apart the blocks would be, where the blocks would be located at the end, and so forth. These tasks were designed to encourage the discovery of the underlying functional relationship. Obviously, in this case each prediction can be tested (unlike the examples earlier). Also, through pattern
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detection, students’ experiences on the function machine may lead to the detection of many qualitative and quantitative properties of linear functions in an intuitive way.

As discussed earlier, prediction relies heavily on construction. We now turn to another typical function task, classification. Classification, however, relies more on interpretation than on construction.

Classification. By classification we refer to actions that involve (a) deciding whether a particular relation is a function (the relation can be represented in any form, e.g., a graph, an algebraic rule, an arrow diagram); (b) identifying a function among other relations; or (c) identifying a special kind of function among other functions. The following studies contain classification tasks: Lovell, 1971; Markovits et al., 1986; Marnyanskii, 1975; Schoenfeld et al., in press; Thomas, 1975; Vinner, 1983; Vinner and Dreyfus, 1989. By and large, the studies that incorporate classification tasks share an abstract and formal approach to the concept of function, that is, they investigate the students’ understanding of the formal definition of a function or understanding of special unifying or distinguishing characteristics of various families of functions.

To a great extent, students’ classification of different kinds of relations depends on both the formal definition of a function that they have been taught and the “concept images” that they have developed based on examples to which they have been exposed (Vinner, 1983). (What Vinner refers to as concept image comes close to what cognitive psychologists refer to as schemas.) Most classification studies approach the concept of a function from a modern perspective, which often is referred to as the Dirichlet-Bourbaki concept of function. The modern approach (also called the set approach) defines a function as a special kind of correspondence (or relation) between two nonempty sets $A$ and $B$. This correspondence assigns to each element in $A$ (the domain) one and only one element in $B$ (the co-domain or range). It follows that a one-to-many correspondence is not accepted as a function,
whereas a many-to-one correspondence could be a function. This modern definition
does not require an explicit or obvious rule of correspondence. (The rule of
correspondence may be completely arbitrary.) In addition, the elements of the sets
need not even be numbers (unlike most of the functions that normally are
introduced in the early curriculum). As such, the new definition includes many
functions that were not recognized as such before the 19th century. The notion of
a function developed, beginning in the 17th century, from a quantity associated
with a curve, to an equation or formula consisting of variables and constants, to
more general cases of dependence between two variables. In general, these early
cases of functions could be seen to be more natural. (The following papers include
discussions of the evolution of the concept of function and the different meanings
associated with it: Bergamini, 1963; Buck, 1970; Eves, 1983; Freudenthal, 1982;

Classification tasks often are used for studying students' concept images of a
function. In most classification tasks, a relation (in an algebraic form or an arrow
diagram) or a graph is presented and the student is asked to classify it as an example
of a function or a nonfunction (e.g., Thomas, 1975). The relations that are presented
in these tasks often follow the modern definition of a function (e.g., a noncontin-
uous graph, or a split-rule) but would not have been regarded as functions in the
early era when the concept of function had just originated (Markovits et al., 1986;
Thomas, 1975; Vinner, 1983; Vinner & Dreyfus, 1989). Thus, such classification
tasks could provide insight into some connections between the historical evolution
of the notion of a function and the development of this concept among naive
students (see section on student learning).

Translation. A third type of task that we consider typical of the domain of
functions and graphs is translation. By translation we refer mainly to (a) the act of
recognizing the same function in different forms of representations; (b) identifying
for a specific transformation of a function in one representation its corresponding
transformation in another representation (Kaput, 1987b, 1987c; Yerushalmy,
1989); or (c) constructing one representation of a function given another one. Thus
translation tasks can rely primarily either on interpretation or on construction.
Translation tasks are fundamental to the concept of function, which “has both
strong graphical and strong nongraphical aspects” (Vinner & Dreyfus, 1989, p.
357). Figure 5 is an example of a translation task taken from Yerushalmy (1988).

This matching task requires moving back and forth from the graphical represent-
tation to the algebraic one. Each step involves either translating graphical features
into algebraic coefficients (e.g., the line that passes through the origin has a zero y-
intercept, therefore its equation can be only 3, 5, or 8), or vice versa [e.g., the
coefficient of x in equation 8 is negative, which translates into a decreasing straight
line (with a negative slope); since none of the given lines decrease, this equation
does not correspond to any of these lines]. Although this specific task is beyond the
early introduction to graphing that we are focusing on, the general translational
issue is salient.

Most of the studies that include translation tasks focus on the connections
between graphical and algebraic representations of functions (such as in the task in
Figure 5). This type of task lends itself well in particular to the use of graphical
technologies (Demana & Waits, 1988; Dugdale, 1982; Goldenberg et al., 1988;
Heid, Sheets, Matras, & Menasian (1988); Shoenfeld et al., in press; Yerushalmy,
MATCH FIVE OUT OF THE EIGHT GIVEN EQUATIONS WITH THE LINE GRAPHS.
EXPLAIN YOUR CHOICE!

1. \( x = 4 \)
2. \( y = 4 \)
3. \( y - 2x = 0 \)
4. \( y = x + 4 \)
5. \( y = 3x \)
6. \( x = 6 \)
7. \( x = -4 \)
8. \( y = -2x \)

FIGURE 5. A translation task
Note. From Yerushalmy, 1988, p. 7; reprinted by permission.

1988). Other authors who directly address the issue of representations of functions are Janvier (1987b, 1987c), Lovell (1971), Markovits et al. (1986), and Zaslavsky (1990).

Scaling. The fourth kind of task that is typical of the domain of functions and graphs (especially to graphing) is scaling. Scaling tasks require particular attention to the axes and their scales and to the units that are measured. In simplest terms, a student might need to decide, for example, whether the number of units each interval stands for is one, two, three, and so forth, and whether both the \( x \)- and \( y \)-axes should use the same scale. When scales are changed for the plotting of the same function the resulting “picture” can look quite different. The issue of scale becomes more fundamental when using graphical technologies (Goldenberg et al., 1988; Heid et al., 1988; Yerushalmy, 1988). Traditionally, it was common to sketch only one graph for each function, restricting oneself to the boundaries of the paper and choosing a reasonable scale to fit these limits. With the new technologies, however, the learner is able to look at the graph of one particular function through many different “windows.” By windows we mean the part of the screen where the graph is displayed. There are no physical boundaries except for those that the student chooses, somewhat arbitrarily, by setting the dimensions of the window (this allows one to zoom in and out). When taking advantage of this opportunity, however, the shape of the graph keeps changing, depending on the scale. This creates a “conceptual demand” that may affect the kind of mental images a student is able to construct (e.g., What kind of mental image of a quadratic function is likely to be developed when a parabola, through certain windows, might look very similar to a straight line?). It is interesting that scale is an issue when using graphs for scientific data analysis and in computer-based instruction, but usually is not an issue when introducing graphing in mathematics classes. It may be that, because the scale is often assumed or given in mathematics instruction (normally the scale is the same on each axis), it then becomes difficult to use or access later in science classes.
Scaling tasks are connected to unit as well as to scale. Unit is an issue in situated contexts such as science. This is in part because in constructing data from scratch the unit may refer to an intensive quantity such as a rate. The unit also may need to be transformed. Some units are more sensible and meaningful than others, whereas other units are more efficient and produce “better” graphs. [For example, in a reading study the number of words or pages read per time unit per child is meaningful; however, the log of that number is far more plottable (Leinhardt & Leinhardt, 1980).]

Like translation tasks, scaling tasks can encompass either interpretation or construction tasks. Figure 6 is an example (taken from Yerushalmy, 1988) of a scaling task that relies on interpretation. The task requires the identification of two seemingly different graphs (that only differ in their scales) as representations of the same function (i.e., graphs 4.b, 4.c, and 4.d all represent the same function, \( y = 2x \)), as well as the realization that graphs that look the same (4.a and 4.b) are representations of two different functions (4.a is a graph of \( y = 4x \), whereas 4.b is a graph of \( y = 2x \)).

THE FOLLOWING ARE FOUR GRAPHICAL DESCRIPTIONS OF LINEAR FUNCTIONS IN SYSTEMS WITH DIFFERENT SCALE UNITS. CHOOSE THE FUNCTION THAT EACH LINE DESCRIBES. EXPLAIN YOUR CHOICE FOR EACH CASE.

**Figure 6.** Scaling tasks

Note. From Yerushalmy, 1988, p. 8, reprinted by permission.
Functions, Graphs, and Graphing

Tasks such as this (Figure 6) address a fundamental visual confoundment that graphical representations carry. This visual confoundment becomes particularly critical with the widespread availability and use of graphical technologies that easily produce many different graphical representations of a single function (Demana & Waits, 1988; Goldenberg et al., 1988; Yerushalmy, 1988). A graph cannot be interpreted fully without taking into account its scales. A complete understanding of graphical representations means realizing what visual features of the graph will not change under the change of scales (e.g., the $x$- and $y$-intercepts) and what features change when the scales are altered (e.g., the geometrical angles that the line creates with each of the axes). Changes of scale is one of the main sources for graphical visual illusions (Goldenberg et al., 1988) that may become obstacles in the process of abstracting from graphs as visual–concrete representations to graphs as symbolic representations.

A scaling task also can rely primarily on construction, such as setting up the axes to plot a graph from scratch (Kerslake, 1981; Krabbendam, 1982; Wavering, 1985). Kerslake, for example, gave students a task that required them to plot on a sheet of graph paper the points $(20, 15)$, $(-14, 3)$, and $(5, -12)$. In this task, the student needs to plan carefully the scaling of the axes to make sure that all the points fit into the coordinate system on the paper and at the same time make sure that the axes follow the conventions, such as, for each axis any two intervals of the same length represent the same number of units. Selecting and constructing suitable scales is an issue particularly in science lab contexts, where a graph describing a situation is plotted based on observations that translate into numerical data. In contrast to the critical role scale plays in interpreting and constructing graphs, scaling tasks rarely are addressed directly in the literature.

To summarize, in this section we discussed two types of actions, interpretation and construction, that are the building blocks of the tasks that are addressed in the literature on graphing and functions. In light of these actions, we analyzed and exemplified four kinds of tasks typical of this domain and inclusive of most of the tasks that appear in the studies: prediction, classification, translation, and scaling. Prediction tasks are essentially construction tasks that build on the patterns that underlie graphs and functions. Classification tasks are mainly interpretation tasks that shed light on the connections between the formal definition of a function and the concept images that surround it. Translation tasks are either interpretation or construction tasks that address the crucial role representations play in the particular domain of functions and graphs. And finally, scaling tasks are also either interpretation or construction tasks that relate to the special conventions, strengths, and limitations of the Cartesian coordinate system. Each of the tasks can be presented in a variety of contexts, which we refer to as situation, the next task variable to which we turn.

**Situation**

Situation encompasses two aspects: the surrounding setting of the task, and the context of the problem. Setting. The first aspect of situation is the setting in which the task is presented, such as a mathematics lesson, a social studies class, or a science laboratory activity (e.g., McKenzie & Padilla, 1986; Mokros & Tinker, 1986; Reed & Evans, 1987).
The vast majority of the studies address the topic of functions and graphs from a mathematics education perspective. (However, there is also a body of work in the science education community that looks into the interactions between the interpretations of physical phenomena and their graphical representations; e.g., Brasell & Rowe, 1989, and McDermott et al., 1987.) A mathematics perspective to the notion of functions and graphs often regards the understanding of the underlying formal and abstract mathematical concepts related to functions and graphs as the main long-term objective. From this perspective, understanding the subtleties of graphs and functions is the ultimate target.

Sometimes the mathematical approach incorporates real-world applications, often taken from the world of science. Rather than serving as the real target, however, these applications often are designed to deepen students' understanding of the more abstract mathematical concepts and perhaps to increase students' motivation by giving relevant and familiar meaning to the problems they encounter. As such, the applications that the mathematics educators provide are not always real problems in terms of how mathematicians or scientists would define a real problem or in the way students perceive them (Lampert, 1989). On the other hand, the science educators’ approach to graphing and functions is quite the reverse. From a scientific approach (we include in this statistics and social studies), graphs serve as representations of real observations and as analytic tools for detecting underlying patterns, which in turn inform the observer and the learner about the phenomena (the target) under investigation. The most common types of tasks associated with the scientific approach to graphing are those of qualitative interpretation, prediction (primarily qualitative), and scaling tasks that require construction (see section on action).

Another kind of setting—one connected more to the mathematics approach—is a game setting. This setting is particularly salient in Schoenfeld et al.’s (in press) work. Schoenfeld and his colleagues followed the underlying ideas of Green Globs (Dugdale, 1982), using a similar environment (Black Blobs) to investigate the processes by which a student comes to grips with the complexity of the notion of a linear function, in both its graphical and algebraic representations. The setting of the function machine that was used by Greeno (1988b) also can be regarded somewhat like a game setting in which students are challenged to come up with appropriate conjectures. Some of the “guess my rule” tasks often have the flavor of a game, too (Davis, 1982).

Context. The second aspect of a situation is the context of the problem itself [often called the problem situation (National Council of Teachers of Mathematics, 1989; Silver, 1988)], which can be more or less contextualized or abstract (what we call an abstract situation is a “naked context,” in Monk’s, 1989, term). The studies that include contextualized tasks often are based on the assumption that it is easier for students to deal with problems that build on familiar situations (e.g., situations they either have experienced or are able to relate to in a meaningful way) than to deal with abstract situations. It is not clear, however, that a real-life kind of context always supports the learning process. For example, some situations have pictorial entailments to them that present the student with difficulties that are tied to the specific cues (see section on student learning).

Two of the more common types of contextualized situations that appear in the literature tend to fall into one of two categories: travel, such as a bicycle traveling
over a hill, a racing car going through a round track, distance-time graphs (Bell &
Janvier, 1981; Clement, 1989; Janvier, 1981b; Kerslake, 1981; Preece, 1983a); or
growth, such as the growth of a bacteria at different temperatures, the average
heights of boys at different ages, the amount of oxygen in the water, the relationship
between the amount of rainfall and crop size, the size of a crowd attending a
demonstration as time changes (Bell & Janvier, 1981; Karplus, 1979; Krabbendam,
1982; Marnyanskii, 1975; Preece, 1983a).

Researchers often design their tasks by selecting extreme cases or particularly
perplexing situations in order to check if students are distracted by irrelevant
features or confused by superficial or visual entailments. Figure 7 is an example
(taken from Janvier, 1981b, p. 115) of a graph representing the speed of a racing

Graph 1.1

Questions

1. Can you tell me, from the graph, how many bends there are along
the track on which the car was driven?

If the answer is "3", we ask "Why?". Otherwise, we help the pupils with a
series of questions each providing a hint and ask after each one the same
question: "How many bends?". Let us give examples:

1(a) This graph is not a picture but...!
1(b) What's the top speed, the slowest speed?,
1(c) What's the speed at 1 km, 2.5 km?
1(d) How many times has the car been slowed down?

In order to prepare the ground for Part C, the following questions are asked:

2. Which bend is the worst? the easiest? the "second worst"?

FIGURE 7. A contextualized graph interpretation task
Note. From Janvier, 1981b, p. 115; reprinted by permission.
car as a function of its distance along the track. In this case, the graph could be (incorrectly) perceived as the actual track the car is driving through, so that students are tempted to interpret the curves on the graph as representing the bends along the track.

Janvier’s (1981b) racing car task is also a good example of the different kinds of influences (both positive and negative) that familiarity with the specific context may have on how successful students are in solving a problem (see section on student learning). Stein and Leinhardt (1989) also found this iconic confusion with graphs in their interviews with fifth graders, but it was uncommon after preliminary graphing instruction.

Most studies place their main focus either on contextualized graphs and functions (Bell & Janvier, 1981; Clement, 1989; Janvier, 1981b; Krabbendam, 1982; Preece, 1983a, 1983b) or on abstract ones (Dreyfus & Eisenberg, 1983; Markovits et al., 1986; Monk, 1989; Schoenfeld et al., in press; Thomas, 1975; Vinner, 1983; Vinner & Dreyfus, 1989; Wagner, 1981b). Few studies focus both on tasks based on a contextualized situation and on tasks that build on an abstract situation (Dreyfus & Eisenberg, 1982; Kerslake, 1981; Markovits et al., 1983; Marnyanskii, 1975; Monk, 1987). The specific situation that is selected for a task dictates to a certain extent the type of variables that are involved in the task.

**Variable**

The notion of a variable is fundamental to understanding many functional relationships and graphical representations. There are several meanings and aspects of a variable that can be discussed (Schoenfeld & Arcavi, 1988). One interpretation of variable is a relatively static one, which emphasizes a variable as a tool for generalization or for describing patterns. (This meaning is the focus of Wagner’s, 1981b, study, as well as of Kuchemann’s, 1984.) This static approach to variable is usually associated with algebraic symbols (e.g., generalized letters). Another interpretation of variable has a more dynamic sense to it, which in essence captures the variability and simultaneous changes in one variable in comparison to another (Janvier, 1981a). The dynamic approach to variable can be represented in a number of ways (e.g., a functional notation, a graph). Regardless of the meaning associated with the notion of variable, little attention is given in the literature to the nature or form of the variables that are connected to the task.

An integral part of a variable is its domain, that is, the set of values that can be assigned to it. The nature of the domain is influenced by the situation. More specifically, the situation suggests the appropriate unit, and the type of unit, in turn, determines the form of the variable. By form we are referring to the property of the unit: categorical, ordinal, interval, or ratio. The first three forms relate to discrete variables, whereas the ratio form is connected to continuous variables. (Janvier, 1983, discusses three types of domains that relate to the form of a variable: (a) a nonordered finite set, which is what we call categorical; (b) a finite ordered set, which could be either ordinal or interval; and (c) an ordered dense set, which is what we call ratio.) In two-dimensional graphs the variables may be in the same form or in different forms. It is interesting that most elementary texts introduce graphing with variables in two forms, for example, day of the week (categorical/ordinal) and number of blue shirts (interval). (This is in part because bar graphs are often the first to be introduced, and the cross between categorical and interval
or ratio variable lends itself well to this type of graph.) Determining the nature of a variable is not always obvious. For example, temperature is a continuous variable by nature. If an experiment is conducted for which the temperature is measured in units of $5^\circ$C, however, it could be perceived as an interval and discrete variable. In this case, the graph representing the experimental data may comprise a finite number of discrete points if it represents only the actual measurements, or it may be a continuous graph if it represents the phenomena as a whole (the latter requires interpolation). The subtleties underlying the nature and representation of the variable seem to influence students’ understanding of discrete versus continuous graphs (see section on student learning).

Context is another characteristic of a variable that is interconnected to the domain. A variable similar to situation can be more or less contextualized or abstract. Abstract variables are usually numerical variables with naked context. They can be either discrete (e.g., the natural numbers, as in Thomas, 1975) or continuous (e.g., the real numbers, as in Vinner, 1983). Contextualized variables, on the other hand, are most often continuous in nature, although they are not always represented as continuous (e.g., Bell & Janvier, 1981, present age as a continuous variable, whereas Bestgen, 1980, considers it discrete, as it is in many textbook examples represented as bar graphs). Of the continuous variables, the following are used most frequently in the literature on graphing and functions: time, length (e.g., distance, height), speed, temperature, weight, and age. Of these, speed is the only intensive variable addressed in the literature. It can be one of the two explicit variables involved in the situation (e.g., Janvier, 1981b), or it can be induced indirectly by interpreting a distance-time graph and figuring out the slope (e.g., Clement, 1985). The latter is considered more difficult and is often subject to confoundment (see section on student learning). Discrete variables are few and are basically counts, such as number of people (Bestgen, 1980; Krabbendam, 1982).

Time is considered a special type of variable that can be interval or ratio (i.e., discrete or continuous). Most contextualized tasks include either a time variable or a time-dependent variable (e.g., age or even temperature). Interpretation and construction of graphs in which one of the variables is time are considered relatively easy. This is because the variation associated with time comes naturally and does not need special attention (see section on student learning). Variables that are not time dependent are rare, one exception being Janvier’s (1983) example of the amount of bacteria obtained from a culture after a specific period of time but at different temperatures.

The characteristics of variables that we have discussed, namely the form of the variable and its context, are not generally addressed in either instruction or in research and are left for the student to induce. Attention to these characteristics is in part a matter of focus.

Focus

As mentioned in the introduction, focus refers to the location of the attention within a specific task, for example, the direction of the graph or the specifics of the axes. Focus also can shift from one location to another, for example, find point $(a, b)$ and then interpret $(a, b)$ on the $x$-axis. Some tasks emphasize the function and the paradigmatic shape (e.g., parabola) or its change, whereas other tasks focus on internal components within the graphical space (e.g., interpret a value where a
point has not been located). Some tasks focus more completely on identification and manipulation of the coordinates. Focus deals with issues of attention and display. For example, in graphs the attention almost always is drawn to intercepts and relations to the origin—instruction gives “landmarks” to the essentially featureless aspects of graphs by pointing out things like intercept or overall shape. But these can be misleading if focus is drawn to another part of the graph.

Focus is a critical issue particularly in learning graphing and functions because there is a wide variety of places to focus on. There are usually few cues, if any, that assist the student in searching for the relevant and most appropriate location of focus. The location of the focus depends largely on the task itself. For example, when qualitatively interpreting or constructing graphs the focus is mainly on the graph itself, rather than on the specifics of the axes. On the other hand, approaching graphs quantitatively often requires more detailed focus on the axes and the quantities they display than on the graph. Although examining local features limits the boundaries within which to focus, global features require a much wider, and often hard to define, space of focus. The four kinds of tasks discussed in the section on action (i.e., prediction, classification, translation, and scaling) could be seen to be associated in general with different locations of focus. Prediction tasks require focus on the visible part or given part of the function or graph that is pointed to in the problem with respect to what is missing (either missing points or an unknown pattern). Classification tasks require focus on the equation or graph as a whole with respect to a known criteria (e.g., a definition). Translation tasks draw the focus from a particular feature of a function in one representation to the same feature in another representation (e.g., y-intercept of a graph and \( b \) in the equation \( y = mx + b \)). Scaling tasks are connected directly to the axes, and they require the focus to be drawn almost entirely to the coordinate system.

To summarize, the complexity of our discussion of the task with respect to the four constructs of action, situation, variable, and focus reflects the complexity of the domain of functions and graphing in terms of structure, difficulties, and kinds of demands it presents both to students and to teachers.

**Student Learning**

We turn now from the specifics of the task to learning by the student. This section is organized into two main parts that represent two aspects of student learning: (a) intuitions, and (b) misconceptions and difficulties. Within each of these sections, we report empirical findings and theoretical discussions that relate to both student understanding of graphs and to student understanding of functions. We begin with a review of student intuitions.

**Intuitions**

We define intuitions as features of students’ knowledge that arise largely from everyday experience, although in the more advanced student they may involve a mixture of everyday and deeply understood formal knowledge. Intuitions are self-evident (and hence tend not to be examined in the sense of an appeal to authority) and are expressive of a fairly cohesive system of beliefs. (We recognize that mathematicians have “intuitions” about their mathematics; we do not feel our definition is inconsistent with theirs, but such intuitions are to date largely understudied.)
The vast majority of work that relates to students' intuitions in this area focuses on functions rather than graphs. Many studies contain findings that relate to students' unschooled ideas, but those findings are not always discussed under a label of intuitions. On the other hand, some studies that are explicitly labeled as investigations of students' intuitions do not, on detailed examination, really deal with students' natural understandings. In general, intuition is taken to mean different things by different authors; hence one cannot expect to find an isomorphic mapping from our definition of intuition to other discussions of intuition in the literature.

Functional relationships. This section discusses intuitions that are related to two primary ways of defining functions: as covariation between two variables and as a correspondence between two sets.

Piaget conducted the most exhaustive research on early understandings of the concept of function. His work supports the existence of an implicit understanding of function as early as age 3½. In a series of studies (Piaget et al., 1968/1977), Piaget and his colleagues explored origins of qualitative functional reasoning in young children and the gradual transformation of such reasoning into quantitative functions. These studies form the basis of Piaget’s position that a “logic of functions” exists in preschoolers and that this early functional logic is derived from the actions of the child. According to Piaget, a young child acquires general schemes of actions that modify an object \( x \) into \( x' \). For example, smashing a ball of clay (\( x \)) with the palm of one’s hand modifies it into a flat piece of clay (\( x' \)). At some level, the child begins to comprehend and expect a regular empirical dependency—in this case, between the resulting flatness of the clay and the force with which it is pressed. Piaget showed how the thinking of preschool children can be characterized by various one-way mappings such as these—mappings that, although functional in nature, are essentially qualitative.

Case (1982; Case, Marini, McKeough, Dennis, & Goldberg, 1986) and Halford (1982; Halford & Wilson, 1980) further investigated the development of functional thinking during early and middle childhood. By and large, these studies agree that by 5 or 6 years of age, children comprehend the nature of functional dependencies in the sense that they solve problems that embody these relations in a variety of conceptual domains (Davidson, in press). In Piaget’s theory, development to more mature forms of functions occurs in several stages, the last of which is characterized by the ability to understand the logical relationships among the four terms of a proportion (i.e., \( a/b = c/d \)). Although Piagetian studies comprise the most detailed work on intuitions, it should be noted that this work (a) has focused solely on a special type of linear functions (in which laws of proportion govern), (b) does not include graphs, and (c) does not attempt to draw ties to issues of pedagogy. By and large, the work has been laboratory based and theoretically driven.

Greeno and his colleagues are building on the positive expectation of significant intuitions about functions that Piaget’s work suggested. One goal of their research is to describe the implicit understandings that older (ages 12-15) students have about functions and variables and to characterize the nature of the situations in which such implicit understandings are effective (Greeno, 1988a). Although Greeno labels the student knowledge that his work describes as implicit understandings, the knowledge possesses the features that we use to define intuitions.

In Greeno’s research, pairs of students were interviewed and asked to reason
about the operation of a machine that embodied linear functions (a version of the
machine used in Piaget’s work on proportions). The machine was a 36-inch board
with two grooves running lengthwise. A ruler marked in inches lay parallel to each
groove. Each groove held a small metal block with a pointer on its top surface. A
string was tied to each block, and at its opposite end the string was attached to an
axle that was at the end of each groove. The axles were attached to wheels with
handles so that they could be turned. A variety of metal sleeves of different
circumferences could be placed on the two axles. In addition, the two axles could
be linked together so that they turned simultaneously, or they could be separated
so they could be turned independently. The components of qualitative analysis for
the student are location (i.e., start and finish), turns of the spool, size of the spool,
and speed. Students were asked a variety of questions including ones of distance
(differences in location between the two spools) and rate (turns per spool). Consid-
ering the formula \( y = mx + b \), \( y \) is the location of the block after the handle was
turned some number of times; \( b \) is the location of the block at the start of counting
the turns; \( m \) is the circumference of the spool; and \( x \) is the number of times the
handle was turned.

Most of the interview questions aimed to uncover students’ intuitions in a
relatively sophisticated way in that they involved the comparison of two functions
rather than inferences based on just one (i.e., comparing movements of block A
with movements of block B). According to Greeno, pilot data indicated that
students found questions about individual functions obvious and were more
engaged and motivated when asked about pairs of functions. Nevertheless, students’
imPLICIT understandings about a single function included the following: (a) a clear
sense that the number of turns and the final position were functionally dependent;
(b) an appreciation that a given variable (e.g., spool size) can take on different
values (variation); (c) the realization that a smaller value of one variable (e.g., spool
size) can be compensated for by a larger value of another variable (e.g., number of
turns); (d) an appreciation that the starting points (i.e., the intercepts) can be varied;
and (e) an understanding that, for any four variables, the fourth value can be
inferred from the values of the other three. With respect to the comparison of two
functions, students exhibited an implicit understanding of the ratio of the values
of two functions.

These findings corroborate Piaget’s notions about implicit understandings and
expectations about empirical dependencies. Greeno’s work also elaborates on
Piaget’s findings by beginning to explicate the role of situational resources in
reasoning about functional relations. Greeno suggests that the students could
observe the functional relations as structural features of events in the machine
situation, and that their understanding of the relations was supported by the evident
causal relations between turning the handle, winding the string around the spool,
and pulling the blocks along the track. He suggests that situations such as these
could be used to anchor students’ understanding of algebraic notation. (It should
be noted that intuitions based on observations of concrete phenomena also may
contribute to a bias toward causality when defining a function.) Like Piaget’s work,
Greeno’s investigations have been concerned solely with functions (not including
graphs of functions), limited to linear functions, and his research tasks have been
laboratory based.

Children’s natural understandings about correspondence form another intuitive
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basis for the concept of function—in this case, function defined as a correspondence or mapping between two sets. Davidson (in press) has suggested that competence on Piagetian function tasks is related to competence in numerical reasoning. This suggestion rests on the observation that early constructions of quantities appear to depend on function properties inasmuch as one-to-one correspondence is often used to establish the cardinal equivalence of two sets. Along these same lines, children’s counting behaviors represent the establishment of a one-to-one correspondence between the objects being counted and the set of whole counting numbers (Gelman & Gallistel, 1978). Indeed, the counting activity itself expresses a particular function which, given a count word, yields its unique successor (Davidson, in press). This use of one-to-one correspondence entails the idea of single-valued mappings, a defining feature of functions. (Intuitions about one-to-one correspondences may bias individuals toward not accepting many-to-one correspondences as legitimate functions.)

Work being conducted at the Technical Education Research Center (TERC) by Russell and her colleagues (Russell & Friel, 1989) suggests that situations that embody one-to-one and many-to-one correspondences are especially compelling to the 4- to 5-year age group. Russell and her colleagues have designed activities such as contrasting the mappings between the number of noses and number of people in the classroom (a one-to-one correspondence) versus the mappings between the number of hands (or toes) and the number of people (a many-to-one correspondence). Their work and work of others (e.g., Davis, 1982) suggests that the mathematics involved in correspondences is inherently interesting to young children and that children bring to such tasks a clear intuitive sense about the establishment of different types of correspondences.

The work just discussed (Piaget, Greeno, Russell) comprises the empirical research that relates to intuitive understandings about functions. Although a number of mathematics educators (see Bergeron & Herscovics, 1982; Malik, 1980) have suggested a learning sequence that identifies intuitive understandings as the initial or ground level, few have investigated empirically what the nature of those intuitive understandings might be. The theoretical discussion on intuitions emerges in debates on how best to introduce functions into the curriculum. Often, those debates center on which definition of a function (the modern or the older definition) should be used. The definition that is used most widely in textbooks and classrooms today presents a function as a special type of relation or correspondence, a relation with a rule that assigns to each member of set A exactly one member of set B (this is also called the set approach). The main focus is on the mapping of the elements of one set onto the elements of another set, on distinguishing between a function and a relation, on determining inverse functions, and on finding functions of functions. Historically, however, a function has been defined as a rule-based relationship between two interconnected numerical variables. The emphasis was on how changes in one variable (usually called the dependent variable) correlated with changes in another variable (usually called the independent variable).

Malik (1980) and others have argued that these two definitions (the old and the modern) represent two very different frames of thought and that it is not obvious how understanding one definition helps to understand the other, particularly at the elementary level. A pervasive concern with the modern instructional definition is that it is abstract and lacks a familiar, situation-based context. (Intuitions about
correspondences are not commonly cited in this debate.) As such, it has been suggested that exposure to this formal, instructed definition of functions fails to build on any intuitive understandings that students possibly may have regarding what a function is (Herscovics, 1982; Janvier, 1983; Malik, 1980; Markovits et al., 1986; Marnyanskii, 1975). Children, it is argued, possess intuitive ideas about functional relationships that have been developed through global perceptual observations of the physical phenomena that surround them every day (e.g., the change of temperature over time). Features of this intuitive knowledge base include notions about dependence, causality, and variation. Children notice that certain things “go together” in the real world and expect that things will change over time and that there will be some pattern to that change. Whereas these childhood notions of functions are based on an implicit sense of variables that have the attributes of being concrete, dynamic, and continuous, the formal definition of function is “algebraic in spirit, appeals to a discrete approach, and lacks a feel for variable” (Malik, 1980, p. 492).

Graphs. The studies discussed in this section differ from those discussed earlier in two main ways: They include intuitions about graphs as well as about functions, and they have been conducted with an eye toward issues of teaching and learning the school topic of functions and graphing. Janvier (Bell & Janvier, 1981) was one of the first to suggest that most graphing instruction was overly focused on quantitative, abstract, localized skills. Rather than begin with tasks that require students to read and plot individual points on scaled Cartesian coordinate systems, Janvier argued that students first should be introduced to qualitative graphs of concrete situations and asked to view them globally instead of pointwise (see Figure 8).

In other words, students should be encouraged to attend to the entire graph as an expression of the relationship between two simultaneously changing variables and to express that relationship in words rather than numbers. This more qualitative approach has the clear advantage of recruiting students’ common sense, intuitions, and reality-checking strategies (Goldenberg, 1987). For example, a student who is interpreting graph B would draw on his or her knowledge that time only increases, that a burning candle undergoes a transformation in appearance over time, and that a burning candle does not rebuild itself. Hence students’ everyday knowledge of real-world events serves as a basis for learning how to interpret a graph of a function.

Although instructional materials based on this approach have been developed (Dugdale & Kibbey, 1986; Swan, 1980), few studies have specifically examined the role that intuitions might play in the reasoning processes that students use as they engage in these types of tasks. One finding that has been replicated is that students deal much more competently with graphs of functions in which one of the variables is time or time-dependent (e.g., temperature, age) (Janvier, 1981b; Krabbendam, 1982). The familiarity of time plus its unidirectional nature (time only increases) seem to account for this. Janvier has suggested that different cognitive processes are needed to deal with functions in which one of the variables is time. In essence, the student does not have to coordinate the interrelationship of two simultaneously changing variables. Rather, he or she only needs to apprehend how one variable (the nontime variable) varies; the time variable is taken into account implicitly. Krabbendam (1982) further supports this notion with his investigations of how
FIGURE 8. Examples of pointwise (top) versus global (bottom) graph interpretation tasks

Note. Top graph from Bolster et al., 1978, p. 345; bottom graph from Dugdale & Kibbey, 1986; reprinted by permission.

secondary students (as a group) constructed qualitative graphs that reflected broad insight into relations that appear in reality (e.g., the number of passengers on a commuter train as a function of the time of day). He observed that time as a variable (either implicitly or explicitly) aided understanding. In addition, when the second variable could be represented vertically (e.g., length, height), the students' ability to construct accurate graphs was enhanced.

These findings on enhanced student performance on time-based graphs seem to support empirically the notion that intuitions that are based on students' knowledge of real-world situations operate successfully when reasoning in the graphing domain. Students draw on their everyday experiences of "things changing together" as a basis for interpreting the graphs. Unequivocal enthusiasm for the successful operation of these intuitions, however, must be tempered by other work on the use of situations in graphing that suggests that sometimes students become distracted by graphical features that correspond on a pictorial level with aspects of the situation (Janvier, 1978; Kerslake, 1977, 1981; McDermott et al., 1987; Preece, 1983a, 1983b; Schultz, Clement, & Mokros, 1986; Stein & Leinhardt, 1989). In addition, they are sometimes overwhelmed by a wealth of situational knowledge that proves to make "graph abstraction" more difficult. These findings (which occur on both time and nontime graphs) are discussed in detail in the section on misconceptions, which follows. Review of the literature on students' learning of functions and graphs reveals more emphasis on documenting the kinds of difficulties and misconceptions
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that students exhibit than on confirming and describing the nature of their intuitions. We turn now to a discussion of misconceptions and difficulties.

Misconceptions

For this review, misconceptions are defined as incorrect features of student knowledge that are repeatable and explicit. Misconceptions in the area of functions and graphs have a different flavor from those that have been documented in the science literature. Whereas science misconceptions often originate in children’s observations and interpretations of real-world events, misconceptions of functions and graphs often are intertwined with previous formal learning. For example, the function concept may be limited because of a lack of variety of instructional examples, or a translation may be performed inaccurately because of confusion over symbolic notation. Schoenfeld and his colleagues (Schoenfeld et al., in press) provide an extended, fine-grained analysis of an advanced student’s learning in this domain in which basic confusions over coordinate system notation were found to be connected to a wide variety of misunderstandings—misunderstandings that were manifested at several structural levels.

We discuss difficulties hand in hand with misconceptions in this section. Difficulties are reported throughout the literature on functions and graphs as they relate to particular features or kinds of functions and graphs that are associated with learning problems. The findings that relate to difficulties do not necessarily propose a misconception as the reason for the difficulty. Rather, they are more likely to imply that something about the task makes it especially difficult.

Misconceptions and difficulties will be discussed under the following subheadings: (a) what is and is not a function; (b) correspondence; (c) linearity; (d) continuous versus discrete graphs; (e) representations of functions; (f) relative reading and interpretation; (g) concept of variable; and (h) notation.

What is and is not a function. Several studies have suggested that students possess inaccurate ideas of what graphs of functions should look like (Lovell, 1971; Markovits et al., 1986; Vinner, 1983; Vinner & Dreyfus, 1989). Most of these findings emerge from classification tasks performed within the modern definition framework and suggest that students have an overly restricted view of the forms that graphs of functions can take. Often students identify only graphs that exhibit an obvious or straightforward pattern as graphs of functions. Lovell (1971) found that some students demanded a linear pattern before recognizing a graph as a graph of a function. In other studies (Vinner, 1983; Vinner & Dreyfus, 1989), students have been found to display more tolerance for a variety of patterns other than linear but still demanded some form of “reasonableness,” such as symmetry, persistence, always increasing or always decreasing, and so forth.

In many cases, students may “know” the accurate, formal definition of a function (i.e., a correspondence between two sets that assigns to every element in the first set exactly one element in the second set), but fail to apply it when deciding whether or not a graph represents a function. Vinner (1983) suggests that in these cases the student’s knowledge of the formal definition is split from what he terms the student’s “concept image”—a concept of what a function is that has been developed through experience with examples of functions. The majority of examples that students are exposed to are functions whose rules of correspondence are given by formulas that
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produce patterns that are obvious or easy to detect when graphed. Hence, students develop the idea that only patterned graphs represent functions; others look strange, artificial, or unnatural.

Along these lines, it should be noted that when classifying only regular, patterned graphs as graphs of functions, students are actually discriminating in a manner that is consistent with historical (and perhaps more natural and intuitive) understandings of what a function is. The modern set-theoretic approach expands the definition of function to include many correspondences that were not recognized as functions by previous generations of mathematicians (Malik, 1980; Vinner & Dreyfus, 1989). These include discontinuous functions, functions defined on split domains (i.e., by different rules on different subdomains), functions with a finite number of exceptional points, and functions defined by means of a graph. The graphs of these newly recognized functions may not follow regular, symmetrical, or easily recognizable patterns. With this historical perspective in mind, students' failure to recognize "unusual" graphs as graphs of functions may be seen in a different light—perhaps not so much as a misconception as a "missed" concept.

In addition to limited conceptions of what constitutes a graph of a function, incomplete notions of what constitutes a function (in any representation, not necessarily a graphical one) have been reported. In some cases, these findings parallel the findings on graphs of functions: Students are most apt to exclude those functions that would not have been so classified under the old definition. For example, students often say that the following functions are not functions: functions composed of an arbitrary correspondence, functions given by more than one rule, and functions not officially recognized and labeled as functions by mathematicians (Markovits et al., 1986; Vinner, 1983; Vinner & Dreyfus, 1989).

In other cases, students' difficulties seem to reflect misunderstandings that do not necessarily relate to preference for the more natural functions defined by the old definition. For example, both Marnyanskii (1975) and Markovits et al. (1986) report that students often assume that functions must consist of changeable quantities. In other words, constant functions are not viewed as legitimate. Several interpretations are possible for these findings. Working within the old definition framework, students might be following a conception that is built on the notion of two things changing together (i.e., their covariation is stressed) thus creating what Marnyanskii calls a "variable hypnosis." Another explanation is that some textbooks present the concepts of variable and constant as opposites, and then proceed by calling the variable a function (Marnyanskii, 1975). Finally, students may be disallowing many-to-one correspondences. This is consistent with other studies (Lovell, 1971; Thomas, 1975; Vinner, 1983; Vinner & Dreyfus, 1989; in addition to Markovits et al., 1986, and Marnyanskii, 1975) that have noted student difficulty with many-to-one correspondences (to be discussed in the next section). Indeed, a constant function is the most extreme form of a many-to-one correspondence, because all of the elements of one set map onto just one element of the other set.

Another expectation students sometimes have regarding a function is that it embodies causality (Marnyanskii, 1975). In many cases, it seems as though students are equating the concept of dependency with the concept of causal connection. This is most apt to occur when students have been exposed primarily to functions that relate to concrete content from everyday life; it may grow out of an overreliance on intuitions.
Correspondence. Two kinds of difficulties have been reported with respect to student understanding of the allowable types of correspondences that constitute functions: the belief that functions must embody a one-to-one correspondence, and confusion between many-to-one and one-to-many correspondences.

Students often require that the elements of two sets be in a one-to-one correspondence before they will claim that conditions have been met for a functional relationship (Markovits et al., 1986; Marnyanskii, 1975; Thomas, 1975; Vinner, 1983). Vinner (1983) suggests that such a condition may result from an implicit requirement for symmetry. In other words, students believe that if each $x$ value maps to one and only one $y$ value, then the reverse should also hold, that is, each $y$ value must map to one and only one $x$ value. This limitation of functions to those in which elements are in a one-to-one correspondence also may be related to the typical instructional sequence. Frequently, the first type of function that students encounter is a linear function, which (with the exception of a constant function) is a one-to-one correspondence. Due to lack of additional examples that embody many-to-one correspondences, students may come to believe that a requirement of functions is that each $y$ value must map to one and only one $x$ value.

Another area of confusion involves distinctions between many-to-one and one-to-many functions (Lovell, 1971; Markovits et al., 1986; Thomas, 1975). Lovell has suggested that a possible reason for this confusion is that, on arrow diagrams (the representation most often used when exemplifying correspondences), students sometimes count the arrows rather than the elements in the set. For example, in Figure 9, they may note that there is one arrow leaving each member of the domain.

![Figure 9](image-url)

**FIGURE 9.** An arrow diagram representation of a many-to-one correspondence that could be misinterpreted as a one-to-many correspondence.

Note. From Lovell, 1971, p. 22; reprinted by permission.
but that many arrows are arriving at a single image, thereby inaccurately labeling the correspondence one-to-many.

**Linearity.** Several studies have pointed to students' tendency to gravitate toward linearity in a variety of situations. Lovell (1971) found that students had a strong tendency to define a function as a relation that produced a linear pattern when graphed. Similarly, Markovits et al. (1986) found that, when asked to generate examples of graphs of functions that would pass through two given points, students produced mostly linear graphs. This tendency toward linearity also has been displayed on tasks that focus on more than two points (Dreyfus & Eisenberg, 1983; Markovits et al., 1983). For example, Markovits et al. showed students several points on a gridwork and asked them to draw a graph to connect the points. In both contextualized and abstract situations, students tended to connect each two consecutive points by a straight line. Furthermore, when queried as to whether other graphs might pass through the points, most students indicated with considerable certainty that the straight graph they had drawn was the only graph that would pass through the points. (Similar responses were given in Markovits et al.'s 1986 study in which students drew straight graphs through two given points.) The students' rationale was that only one line can pass through any two given points, suggesting that they were overgeneralizing the special property of linear functions, that is, that a line is uniquely determined by two points.

Students' tendencies toward linearity also have been reported on interpolation tasks that were designed to require curved interpolation. Asked to interpolate between two data points that were generated by a hypothetical science experiment (see task section, Bacteria Puzzle), students tended to connect the two given points with a straight line (Karplus, 1979). Similarly, asked whether or not a new point (one located on a straight line between two given points) fell on the graph of a parabola, students tended to state incorrectly that the new point was on the graph of the parabola (Zaslavsky, 1987).

Strong pulls toward features of linearity also have been shown on translation tasks. For example, Zaslavsky (1987) asked students to find an equation for a graph of a parabola that contained three labeled points. Some students used only two points to find the equation. Furthermore, they used them in a manner that suggested a linear mindset, that is, they used them to calculate what would be the slope of the straight line through those two points. Then they inserted this "slope value" into the parabola form of an equation as the leading coefficient.

We might speculate as to why students display this excessive adherence to linearity. In the study by Karplus (1979), some students justified straight lines because they were "more accurate." This belief in the intrinsic accuracy of straight lines was corroborated in a different way by another group of students in the same study who correctly interpolated using curved lines, but for the wrong reason. They said that they did not use straight lines because straight lines were too exact to represent the graph of a natural process, in this case, bacteria growth.

Students' tendency to default to straight line graphs also might have been encouraged by the popular preschool activity of "connect the dots." In this activity, students receive a great deal of practice joining contiguous dots with straight lines. It should be noted that such an activity is also common in elementary graphing lessons in which the plotting of points on the Cartesian coordinate system is being introduced or reviewed through coordinate geometry and where the final "picture"
is the reward for the exercise. Finally, students' tendency to revert to linearity also might be explained by the fact that the first family of functions to which students are usually introduced is linear functions. Later, after being exposed to other families of functions, students may still exhibit a tendency to overgeneralize the properties that they learned in conjunction with linear functions. (Another property that students overgeneralize is one-to-one correspondence.) It is interesting to note that, outside the area of functions and graphs, Matz (1982) also has identified the overgeneralization of linear properties. Her work suggests that reciprocal, cancellation, and generalized distribution errors in algebra were based on assumptions that linear properties applied when the context indicated they were inappropriate.

Continuous versus discrete graphs. Deciding whether a graph is or should be represented in a continuous or a discrete manner is not trivial. The literature contains numerous examples of students erring in both directions, that is, representing or interpreting continuous data in a discrete manner and representing or interpreting discrete data in a continuous manner. Both are discussed in this subsection.

Students have been shown to be confused about the meaning that can be attributed to the uninterrupted line of a continuous graph. Several studies have pointed to students' tendency to "see" only the marked points on such a graph (Janvier, 1983; Kerslake, 1981; Mansfield, 1985; Stein & Leinhardt, 1989). For example, Kerslake presented students with a set of ordered pairs and instructed them to plot the points and connect them with a straight line. Then they were asked a series of questions about points that lay on the graph but were "unmarked." Several students said that there were no points between two marked points. Other students thought there was just one, presumably the midpoint. When asked how many points lay on the line altogether, several students gave the actual number of points that they had plotted, others gave the number of whole number points, still others counted the number of locations where the line crossed the grid. Even those students who were aware that very many points lay on the graph seemed to limit their conception of the total number by the physical constraints of actually drawing the points. This relates to Mansfield's (1985) finding that students often interpret points as having a physical mass rather than as abstract entities. When viewed as dots with density, students believe that the number of points that would fit between two marked points must be bounded by the dimensions of size and space.

Students also have been shown to connect discrete points when it is inappropriate to do so (Kerslake, 1981; Markovits et al., 1986). Kerslake presented a scattergram that consisted of four points representing the waist and height measurements of four children and asked students if the points should be "joined up." Some students clearly believed that the points should be connected, mostly for reasons that related to their ideas about the appearance of a graph. For example, some students said that joining the points would make the graph more accurate and neater; others said the points should be joined only if they lie in a straight line or make some other recognizable pattern. These students failed to realize that there would be no meaning attached to the points on the line that fell between the four initial marked points.

In general, then, students often maintain a strict focus on individual points whether or not they are connected with a line. In other words, although lines are accepted as a legitimate part of graphs, they seem to serve a connecting function rather than possessing a meaning in their own right. To some extent, the issue of
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discrete versus continuous representations is intertwined with pointwise versus relative reading and interpretation. Students' tendency to focus on individual points (discussed at length in an upcoming section) may be seen as related to what Janvier (1987b) calls the tendency to “discretize” graphs. Such a tendency is at odds with viewing the graph as an object or as a conceptual entity in its own right (Schoenfeld et al., in press; Stein et al., in press; Yerushalmy, 1988).

In the Markovits et al. work (1986), students' difficulties with the continuous versus discrete issue were interpreted with respect to focus on the function rule and lack of attention to its domain and range. The algebraic function \( f(x) = 4x + 6 \) was presented with the domain and range restricted to the natural numbers. When asked if a continuous graph passing through the appropriate points represented an identical function, half the students said yes, neglecting the fact that only points with coordinates that are the counting numbers were included in the domain and range and therefore only counting numbers should be represented on the graph. Similarly, when presented with a graph on which the appropriate discrete points were marked, most students said it was not a graph of the function; several said that the points should be connected, ignoring the restriction on the domain and range.

Representations of functions. Several representational systems can be used to display a function. These include ordered pairs, equations, graphs, and verbal descriptions of relationships. Janvier (1987d) has called attention to the psychological processes involved in moving from one representation to another. He termed these movements “translations” and noted that such translations have directionality; for example, moving from an equation to a graph involves different psychological processes than moving from a graph to an equation. Although translations among all of these representations of functions have been discussed, most tasks in the empirical literature focus on translating from equations to graphs and vice versa (see task section on translation).

A logical analysis of these two tasks (translations from equations to graphs and vice versa) suggests that movement from graphs to their equations would be the more difficult task because it involves pattern detection, whereas graphing an equation involves, by comparison, a relatively straightforward series of steps, that is, generating ordered pairs, plotting them on a Cartesian grid, and connecting them with a line. In addition, most instruction proceeds in the equation-to-graph direction.

Empirical work in this area supports the notion that moving from a graph to an equation is more difficult. For example, recent National Assessment of Educational Progress results indicate that, when given a ruler and a sheet of paper with labeled axes, only 18% of 17-year-olds produced a correct graph corresponding to a linear equation. However, the reverse translation was even more difficult. Given a graph of a straight line with indicated intercepts \((-3, 0)\) and \((0, 5)\), only 5% of 17-year-olds could generate the equation (Carpenter, Corbit, Kepner, Lindquist, & Reys, 1981). Other studies have confirmed these findings but with different age groups—Stein and Leinhardt (1989) with 10- and 11-year-olds and Markovits et al. (1986) with 14- and 15-year-olds. Markovits et al. (1986) only found translations from graphs to equations to be more difficult than the reverse when the functions were familiar. When the function was less familiar (e.g., piecewise or constant functions), translations in both directions were found to be equally hard. Kerslake (1977, 1981)
also found translations involving constant functions to be exceptionally difficult, probably because one of the variables is “missing.” This finding is supported by Zaslavsky (1988), who found that translations were especially problematic when one of the coefficients of a quadratic function was zero, that is, appeared to be missing.

Some work has investigated translations between more complex graphs and their equations. For example, students also have been observed to prefer translating from equations to graphs on matching tasks involving parabolas. Zaslavsky (1987) asked secondary students to determine which of four parabolas corresponded to a given equation and which of four equations corresponded to a given parabola. In both cases, the students tended to work in the same direction. They used the parameters of the equations as a base to check the graphs, thus moving in an equation-to-graph direction.

In addition to these studies, students’ ability to perform translations between graphical and algebraic representations often are examined with the aid of computer technologies (Demana & Waits, 1988; Dugdale, 1982; Goldenberg et al., 1988; Heid et al., 1988; Schoenfeld et al., in press; Yerushalmy, 1988). Learning to perform translation tasks lends itself well to computer technologies, because graphs can be generated quickly by the computer, freeing the student from the burden of calculating, plotting, and drawing (Schoenfeld, 1988). Thus the student is provided with the opportunity to view many graphs and their corresponding equations and can begin to examine the relationship between graphical entailments and algebraic parameters (e.g., that the steepness and direction of the graph is related to the magnitude and sign of the leading coefficient of the equation of the graph). After exposure to 20 computer-assisted lessons (over 3 months) that emphasized the connection between the algebraic and graphical representations of functions, Yerushalmy (1988) found an average performance of 88% on the matching task shown in Figure 5 (task section). Examination of the students’ explanations, however, revealed some rather unconventional notions. For example, a frequent explanation for matching a function to a graph was the number of quadrants through which the line passed. In addition, when equations were not presented in the $y = mx + b$ format (see no. 3, Figure 5, task section), students had difficulty matching them with their graphs.

Work being done by Schoenfeld et al. (in press) provides an exceptionally detailed examination of the interface in one learner’s mind between algebraic and graphical representations. Within the realm of linear functions, Schoenfeld et al. show how one particular student connected (or failed to connect) various graphical entailments to parameters in the algebraic equation for a straight line, $y = mx + b$. Their analysis revealed several misconceptions that could be traced to a missing “Cartesian connection.” For example, the student’s knowledge and use of the formula for slope was fragile in the sense that its meaning was devoid of any graphical entailments. Whereas a more knowledgeable individual would recognize that $y_2 - y_1$ and $x_2 - x_1$ are directed line segments so that their ratio indicates both direction and steepness, such connections seem to be lacking in the student that Schoenfeld studied. The deep, fine-grained analysis reported by Schoenfeld and his colleagues (in press) can be seen as providing an initial sketch of the cognitive processes that underlie the difficulties students experience when translating between graphical and algebraic representations of functions.
Relative reading and interpretation. In this section, we discuss difficulties related to attempts to construct and interpret graphs that represent situations. A useful feature of many such graphs is that their patterns and shapes highlight features of the underlying situation that otherwise would be hard to detect. Nevertheless, when examining graphs of situations students often restrict their focus to an individual point or group of points as opposed to the more global features of the graph, such as its general shape, intervals of rise or fall, and so forth (Bell & Janvier, 1981). Janvier (1978, 1981a, 1981b) has suggested that this pointwise focus is not surprising given traditional instruction in which students are asked to plot a graph from a table of ordered pairs and then are presented with a series of questions that can be answered by the table alone. Under these conditions, students typically use graphs in much the same way as they use tables—to look up specific pieces of information. Little or no attention is devoted to elaborating the properties of the underlying situation.

The literature contains examples of a host of difficulties and misconceptions associated with students' learning to focus more broadly on the overall shape of the graph or parts of the graph. We have loosely grouped these difficulties under three main categories: interval/point confusion, slope/height confusion, and iconic interpretations.

1. Interval/point confusion: As they interpret graphs, students often narrow their focus to a single point even though a range of points (an interval) is more appropriate. This is most apt to occur when the wording of a question is ambiguous. For an example the reader is referred back to Figure 2. With respect to that graph, Preece (1983b) found that students often responded with a single point when asked the questions, “When are girls heavier than boys?” and “When are girls growing faster than boys?” According to Preece (1983b), students found the word when to be rather imprecise and “took the easiest option and gave a single point as the answer.” Bell and Janvier (1981) noted a similar tendency when students were interpreting a graph that showed the changes in populations of two microbe cultures in relation to their times of feeding. When students were asked when population B was greater than population A, they tended to give the maximum of B instead of the entire interval during which population B was greater than population A.

Given the ambiguity of the word when, it must be granted that technically these students were correct. Nevertheless, their focus on a single point does seem to be part of an overall tendency to interpret graphs pointwise. Another manifestation of this tendency is slope/height confusion.

2. Slope/height confusion: There are numerous examples in the literature of students’ confusing gradients with maximum (or minimum) values (Bell & Janvier, 1981; Janvier, 1978; McDermott et al., 1987; Preece, 1983b). Students have been found to confound these two graphical features on both interpretation and construction tasks. For example, given the top graph in Figure 10 and asked the question, “At the instant t = 2, is the speed of object A greater than, less than, or equal to the speed of object B?”, many students incorrectly answered “less than” (McDermott et al., 1987). This suggests that the students are focusing on the fact that at t = 2 the value of point A is less than the value of point B; they are not attending to the fact that at t = 2 the slope of A is greater than the slope of B.

Janvier (1978) noted a similar problem on a construction task. Students were given the lower graph in Figure 10 and told that A represented a wide jar being
FIGURE 10. Graphing tasks on which students have exhibited slope/height confusion

Note. Top graph from McDermott, Rosenquist, & vanZee, 1987, p. 504; bottom graph from Janvier, 1984, as reported in Clement, 1985, p. 4; reprinted by permission.
filled with water. They were then asked to draw a graph for a narrower jar being filled with water. Instead of drawing the correct graph (C), most students drew graph B. The students took account of the fact that the water would be higher in the narrower jar, but failed to note that the rate (the steepness or slope of the graph) would be greater also.

No consensus exists regarding the cause for such errors (Clement, 1989). A common interpretation is that students are confusing two graphical features: highest value versus slope. On the other hand, McDermott et al. (1987) have pointed out that students have been observed to confound the physical concepts of speed and distance at a conceptual level when tasks have not included a graphical representation. Clement (1989) provides yet another interpretation based on his competence model of the cognitive underpinnings of successful graph interpretation. Within his model, students who state that the speed of object A is greater than the speed of object B (first graph in Figure 10) are successfully isolating the variable of speed, but failing to map it to the correct graphical feature.

3. Iconic interpretation: The most frequently cited student error with respect to interpreting and constructing graphs that represent situations is iconic interpretation. A host of findings support the notion that students sometimes interpret a graph of a situation as a literal picture of that situation (Janvier, 1978; Kerslake, 1977, 1981; McDermott et al., 1987; Preece, 1983b; Schultz, Clement, & Mokros, 1986; Stein & Leinhardt, 1989). A frequently cited finding in this regard is students' interpretation of travel graphs as the paths of actual journeys (Kerslake, 1981). Students were shown the graphs that appear in Figure 11 and asked which of them represented journeys and to describe, for each graph, what happens.

Many students thought that all three graphs could represent journeys, even though on the first graph a traveler would gain distance with no passage of time and on the second graph time would reverse itself. In their descriptions of the journey situations, students said things (for the first graph) such as “Going along, up, and along,” or “Climbing a vertical wall,” or “Going east, then north, then east.” For the final graph, some students said “Going uphill, then down hill, then up again” or “Climbing a mountain.”

Which of the graphs below represent journeys? Describe what happens in each case.

![Graphs](image-url)

FIGURE 11. Graph interpretation task
Note. From Kerslake, 1981, p. 128; reprinted by permission.
Another example of iconic interpretation is provided in Janvier (1981b). When shown the graph in Figure 7, students were first asked how many bends there were in the racetrack. The second part of the study involved asking students to sketch graphs that compared distance and speed for three racetracks whose pictures were provided. The third and final task involved selection of the right track (from a group of seven) for the graph shown in Figure 7. The main mistake in all of these tasks was confusing the graph with the track (iconic interpretation). In the first part, this meant answering that there were 6, 8, or 9 bends. For this first question, familiarity with cars' performance seemed to provide situational support to the boys' ability to reason with the symbolism of the graph. The girls' lack of experience with thinking about cars, on the other hand, was associated with serious trouble, especially for those girls with less developed graphical abilities.

The final task that required students to coordinate a graph with several tracks was challenging for both boys and girls. As Janvier (1981b) noted, on the one hand, the students needed to consider the graph symbolically, that is, pictorial resemblance with elements of the tracks had to be disregarded. On the other hand, the pictures of tracks had to be examined abstractly, that is, at a level that took account of but went beyond driving performance. In this final task, the strong situational appeals that driving race cars had for the boys often made it difficult for them to perform the abstractions necessary to bring their knowledge to the symbolic level of the graph. The nature of this difficulty led Janvier to speculate about the use of situations in graphing specifically and in mathematics education in general:

> The situation which helped many boys in part A now makes the task much harder. Indeed for many boys the vivid memories and/or strong mental images which support their thinking conflict with the more basic abstract aspects of the problem. The necessary abstraction is made more difficult because of the larger basis of information from which it is derived. We are actually provided with several nice examples of how a task can be made more demanding when a situation is called into play. (p. 119)

Janvier has termed distractors based on past experiences personal distractors to differentiate them from visual or pictorial distractors. Indeed, both personal and pictorial distractors can be operating simultaneously, complicating students’ attempts to make sense of graphs.

Clement (1989) has called attention to two kinds of errors that students often make when dealing with graphs of situations: feature correspondence errors and global correspondence errors. As an example of a feature correspondence error he provides the top graph shown in Figure 12.

After generating such a graph, he found students often would state incorrectly that the two cars represented in the graph were passing each other where the graphs intersect. Clement notes that, in this case, students were not drawing whole pictures, but rather were mapping a local visual feature of the problem scene (path intersection) onto a similar feature of the graph (same location). In contrast, global correspondence errors involve matching the shape of an entire problem scene to the shape of the entire graph in a global manner. The bottom graph in Figure 12 was generated when asked to draw a graph of speed versus time for a bicycle traveling over a hill.

With respect to all of the above, it should be noted that some situations are more
FIGURE 12. Examples of a local correspondence error (top) and a global correspondence error (bottom)

likely to bring forth misconceptions than are others. In general, students seem to have a difficult time grasping concepts that arise from variables not actually shown on the graph. They are more likely to exhibit misconceptions, for instance, on a distance–time graph (where speed is a rate “created” by the combination of distance and time) than they are on a graph that explicitly shows the relationship between speed and time (i.e., with time on the x axis and speed on the y axis). Misconceptions also are often evidenced on graphs that contain a pronounced feature, such as a sudden rise or fall or a discontinuous curve, and on graphs whose variables are especially familiar and provocative (e.g., the sewage graph by Preece, 1983a). Indeed, what Clement (1989) calls a feature correspondence error may be related to the finding that pronounced graphical features often distract students.

Concept of variable. Historically, the development of the notion of variable is connected intimately with the development of the notion of function. According to Bergamini (1963), the development of the Cartesian coordinate system in the 17th century “captured and tamed the changing relationships between interconnected quantities” (p. 86) and, in the process, gave rise to the ideas of variables and functions.

If an \( x \) and a \( y \) can be related through an equation or graph, they are called “variables”: that is, one changes in value as the other changes in value. The two have what is known as a functional relationship; the variable whose change of value comes about as a result of the other variable’s change of value is called a “function” of that other variable. (p. 86)

Just as the definition of a function has changed over time, so has that of a variable. The quote from Bergamini stresses the older way of thinking about a variable—a way that captures the dynamic and continuous attributes of variables. More modern notions, on the other hand, depend on the notion of domain. Overall, the idea of variable is subtle, difficult, and far from unitary (Freudenthal, 1983; Schoenfeld & Arcavi, 1988).

One study that empirically examined how students think about variables within the context of a function was by Wagner (1981b). His findings indicated that many students believed that changing the symbol for the variable in a functional equation changed some critical aspects of the function. For example, \( 7 \times W + 22 = 109 \) was viewed as different from \( 7 \times N + 22 = 109 \). These findings suggest that students overly focused on arbitrary symbol substitution but missed the central idea of a functional relationship between two variables. On the other hand, students merely may not note the correct dimension on which the material is similar; they are correct in thinking that travel and financial interest are different even if the equations are similar.

It can be argued that knowledge of the concept of variable is a prerequisite to a full understanding of functions, particularly when such an understanding includes functions defined as the relationship between an independent and a dependent variable. On the other hand, an individual’s understanding of variable may deepen as he or she comes to know functions. Marnyanskii’s (1975) findings suggest that before studying functions, students often see a variable as a “unique object.” (This is probably an accurate view of how Wagner’s subjects viewed variables. Because the letters in the equations stood for unique objects, changing the letters changed the objects.) Only as they work with variables in a functional relationship do they
become aware of “the set of numbers (as a rule, an infinite set) concealed behind it” (Marnyanskii, 1975, p. 168).

Kuchemann’s (1981) work points to the fact that students often can manipulate letters in equations and not have an understanding of variable. One of Kuchemann’s items was as follows:

Blue pencils cost 5 pence each and red pencils cost 6 pence each. I buy some blue and some red pencils and altogether it costs me 90 pence. If \( b \) is the number of blue pencils bought, and if \( r \) is the number of red pencils bought, what can you write down about \( b \) and \( r \)? (p. 107)

The percentage of correct responses were 2%, 11%, and 13% for 13-, 14-, and 15-year-olds respectively. The most common incorrect response was \( b + r = 90 \). Herscovics (1989) interpreted this incorrect response as indicating that students have a tendency to view literal symbols as representing sets, that is, the letters were used merely as labels identifying specific sets.

Notation. In this section, we discuss student difficulties related to the unique notational systems inherent in both the graphical and algebraic symbols that are used to represent functions. We begin with an examination of the kinds of problems students encounter when setting up and moving within the Cartesian coordinate system.

Many studies take for granted students’ ability to construct the axes of the coordinate system. Hence construction of axes is rarely a part of most graphing tasks; the axes, properly scaled and labeled, are provided as givens. Yet, there is evidence that the construction of axes requires a rather sophisticated set of knowledge and skills. For example, Vergnaud and Errecalde (1980) investigated 10- to 13-year-old students’ translation of numerical data onto a straight line. They found that some students represented the data as segments of a line, with the lengths of the segments corresponding to the magnitudes of the numbers. When the data were represented as such, the authors report discovery of the hierarchy shown in Figure 13. As shown on the figure, each drawing becomes progressively more similar to an axis.

Other students attempted to represent the data as points on a line. Many of these students, however, failed to appreciate that for a datum to be shown as a point on a scale, issues of both relative order and measure need to be addressed. Students often positioned dots on a line in their correct relative order but without representing their measure. Furthermore, the distances between the dots often did not correspond to any measured difference between the given magnitudes. In short, many students did not come to grips with the notion of an interval scale.

This study indicated that constructing a singular scale is far from trivial. Results from other studies (Goldenberg et al., 1988; Kerslake, 1981; Wavering, 1985) suggest that students have similar problems when setting up two axes for a Cartesian coordinate system. In this case, however, the problems are compounded with the added requirements of maintaining a one-to-one correspondence between the two sets of data and, in some cases, constructing both positive and negative parts of the axes. In the Kerslake study, for instance, many students thought that it was legitimate to construct different scales for the positive and negative parts of the axes. Goldenberg et al. (1988), on the other hand, found that some students believed that the scales on the \( x \) and the \( y \) axes needed to be symmetrical, even when this was counterproductive to their efforts to make the graph more visually accessible.
End-to-end positioning of segments.

Left-end alignment of segments.

Left-end alignment of segments on the same line, with explicit representation of the starting and end points of each segment.

FIGURE 13. Ways in which students translated numerical data onto a straight line using line segments
Note. From Herscovics, 1989, p. 69; reprinted by permission.

Scaling issues also arise in the interpretation of graphs. The inclination and shape of a graph are, to a great extent, dependent on the coordinate system. Learners need to develop an understanding of which features of a graph are indigenous to the graph itself (e.g., the y-intercept) and which features are responsive to the system on which it is constructed (e.g., the slope of the graph). In this vein, Kerslake (1981) investigated the degree to which students understood the effect that changing the scale of the axes would have on the appearance of the graph. Students were given the graphs shown in Figure 14 and asked which two showed the same information.

The percentage of students who realized that the first two were the same graph was 46.4, 63.4, and 68.5 respectively for 13-, 14-, and 15-year-olds. Using computer-generated graphs, Yerushalmy (1988) reported somewhat better results for eighth graders presented with a similar task (shown in Figure 6). As previously explained, this task includes two different pictures of the same function and two identical pictures of different functions. The percentage of correct responses for figures a through d were 91, 94, 61, and 69.7, respectively. Yerushalmy’s analysis of the types of explanations that students gave for their answers may provide insight into their relatively good performance. On average, four times as many explanations were based on computations as opposed to visual considerations.

Summary

An overview of the variety of misconceptions and difficulties presented in this section reveals learning problems in three broad areas: a desire for regularity, a
pointwise focus, and difficulty with the abstractions of the graphical world. Students have been shown to exhibit a desire for regularity as they search for a recognizable pattern to determine what is and is not an acceptable graph. This is manifested by their unwillingness to consider irregular graphs as graphs of functions, their preference for one-to-one correspondences, their strong tendency to default to properties of linearity, and their tendency to connect points (whether or not it is appropriate) because it "looks better." Students' tendency to focus on individual points also is manifested in a variety of ways: their discretization of continuous data, their lack of propensity to use the pattern of a graph to determine its equation, and their disproportionate emphasis on single points, such as maximum and minimum values, at the expense of intervals and (especially) slope. Finally, dealing with a graph on its own terms, that is as an abstraction (with a considerable amount of convention and notation to be mastered), is not trivial. Failure to apprehend graphs as abstract formalisms results in a number of difficulties, including assigning more meaning to the scale than is mathematically warranted, not comprehending the significance of the slope of a graph as a measure of rate, and viewing a graph as a picture.

Teaching

As we mentioned in the introduction, our conceptualization of teaching is one that includes all aspects of the task of graphing and functions and issues of student learning, but also includes some additional wrinkles. One of the most critical issues is the requirements and effects of the teacher's subject matter knowledge. To some extent, subject matter knowledge can be sketched out only in terms of the other topics that we will discuss in this section: entry to, or starting point for, the topic, sequencing, explanations, examples, and representations. In discussing this section we must again qualify our data base of articles. Just as with the topic of intuitions, for which we often inferred a role where there was no research, we must in this section construct aspects of teaching that are not present in the literature; indeed of the many articles we reviewed, almost 75% had an obligatory section at the end called something like "implications for teaching," but few dealt directly with research on the study of teaching these topics.

For some, discussion of teaching is synonymous with discussion of subject
analysis, situations of use, and student learning. We believe, however, that teaching is more than the logical extension of facilitation of learning by the individual—it involves guiding and presenting. Guiding and presenting, in turn, involve selecting and transforming the teacher’s own knowledge of graphs, graphing, and functions and other mathematical ideas as well as the supporting system of knowledge from text (be it printed or computer based). The teacher’s subject matter knowledge empowers the teacher with the confidence and capability to make interconnections, build analogies, create examples, take intellectual excursions, and point toward future use and interrelationships. It also produces a type of stability in presentation and response (Leinhardt, Stein, & Baxter, 1988). Limitations on subject matter knowledge, on the other hand, often reduce the flexibility and creativity of a teacher as well as create a kind of authoritarianism toward the subject and student that permits little or no exploration of ideas (Shulman, 1986; Grossman, 1987; Grossman & Gudmundsdottir, 1987). (This, however, is not the only cause of such behavior.) To put graphing and functions into context within a larger framework or to draw on unanticipated sources of student knowledge, the teacher must know the content domain extremely well. Several studies, however, seem to point to a limitation on that knowledge (Even, 1989; Stein et al., in press). One of the goals of teaching is to help the student establish powerful concepts in functions, graphs, and graphing. We briefly summarize four.

Functions establish the conformation of particular relationships between changing entities, and graphs help to display selected portions of the relationship. But the features, or landmarks, of graphs or the graphical world are not isomorphic to either the data or the algebraic worlds they represent. Arbitrarily instantiated lines form the axes, and where a line or a curve crosses one (the y-intercept) is detachably concretized in one algebraic representation (the expression for a line), but the equally graphically salient point on the other axis goes unmarked. Conceptualizing point plus slope as line rather than point to point as line is a task for the teacher. Notice that for the purpose of constructing a drawing, point to point is easier.

The linkage of algebraic formula (slope) to its graphical instantiation is another requirement. This is often difficult because the path of a line is what is salient visually, not the relationship of relative changes to changes. Decomposing slope into relations of absolute values (moving from four numbers to one) produces the magnitude of slope, and sign yields direction; these mappings, so absent in Schoenfeld et al.’s (in press) subject IN, are surely an important hook for the teacher to install.

Directionality in the Cartesian system is a third concept or, perhaps more precisely, less a concept than an orientation. Teachers must be aware of the residing conflicts of absolute value and sign that are so visible in graphs. Values get bigger as the focus moves up and to the right. This is convention. Values get smaller to the left and down. However, competing with this vision is both the sense and term of origin, where again the two unmarked (algebraically unmarked) axes cross. Things increase in absolute size as they move symmetrically out from the origin—two features of magnitude (size and direction) must be tracked. The teacher must have a sense of this conflict to point it out to students.

Finally, a central issue is the connection of informal, intuitive, and qualitative functions to the more explicit and formal ones. This last issue is the one most
Focused on by research in the area. It is most salient in discussions of how to begin the teaching of functions and graphing.

Entry. Or, where to start? The selection of a beginning point, or entry, into a topic influences how the rest of the material will be presented, sequenced, and interwoven. To some extent, all points of entry have their limitations; the object is to craft the introduction, and later sequencing, in ways that enhance the early understanding and limit the misunderstandings that may have developed. In a topic as intricate and far reaching as functions, graphs, and graphing, the issue may well be one in which there are multiple beginnings—one for graphing, one for functions, one for data recording, one for qualitative interpretation, and so forth. Or put differently, there may well be a core problem that serves as the basis or link by which all of the specifics are connected or attached.

By the time a student is ready to deal with the topics of functions or of graphs, he or she has had some mathematics education and often some experience with both algebraic and pictorial representations of quantities. The entry point to the topic is not totally obvious. Because the authors of various theories and research that we have considered have not been asked directly, How should one start the topics of graphing and functions?, it would be unreasonable and perhaps misleading to put forward three competing views on how the topics should be introduced. There are, however, at least three trends in emphasis that could be considered candidates for beginning the topic: (a) Discover the rule; (b) generate the data and plot; and (c) interpret qualitative graphs of situations. The already-mentioned issue of which definition of function to start with is salient here as well. Textbooks, which are the supporting backbone for most teachers (whether or not one believes this should be the case), present the teacher with essentially two distinct paths: Guess My Rule (a), or learn the conventions of graphing (a particular version of b). In some cases, texts merge the two approaches, interweaving different types of graphs and tables of ordered pairs for linear graphs (Stein & Baxter, 1989). Texts approach the conventions of graphing in two ways: One is to graph geometric shapes or simple figures (e.g., triangles, stars, hearts) from pairs of values which are "altered" (moved, enlarged, shrunk) by systematic changes in the values; the other is to construct linear graphs from tables of ordered pairs.

We found no research that contrasted the effects of starting this topic from one or another angle. In our exploration of different texts, one starting with function machines (Open Court), the other starting with plotting (Scott Foresman), we saw no effects (Stein & Leinhardt, 1989). Therefore we will lay out the rationales for each of the different approaches to beginning the topics rather than contrast their efficacy. Guess My Rule (see Davis, 1982; Willoughby et al., 1981) has students build up their intuitive sense of rule and functions without starting from a formal constraining and detached view of what a function is or is not (Davis, 1982; Herscovics, 1982; Janvier, 1982). Essentially, the strand of a function is developed independently from the strand of a graph (to be linked up at some later point). It is worth noting that starting from this perspective does not require that the student have a command of the notational conventions to participate in a dialogue about the rules. Further, from the point of view of the teacher operating without direct guidance from a text, it is quite easy to construct the examples for the game. The teacher simply constructs a rule that he or she does not publicly share (e.g., plus 3,
or plus 3 times \( \frac{1}{5} \) and then plays with students by having them give start or input values and then supplies the appropriate second, or output, value. Both inputs and outputs are recorded in some systematic fashion, usually a table. Knowing how this attaches to the more elaborate curriculum is important, but a lack of such knowledge at this entry point may be of minor consequence.

Approaches that build on the generation of data from the world (e.g., plant growth charts in science, height and weight charts, etc.) tend to combine some of the aspects of Guess My Rule with aspects of the learning of graphical conventions. The activity is to take data gathered in a real context, organize them in some fashion, and build a graph. Such approaches can emphasize either aspect. But they are clearly construction-to-interpretation tasks. The demand placed on the teacher to select the “right” problem is more burdensome than for Guess My Rule. Further, it requires a deeper subject matter knowledge for the teacher than Guess My Rule, because issues of data management, scale, unit, selection of axes, and interpretation of lines in terms of functions all come into play. An advantage to this point of entry is that it links very well to other aspects of the curriculum, especially science and social studies, which often contain graphical work but not work with functions (Janvier, 1978; Kerslake, 1981; M. Lampert, personal communication, 1988; Swan, 1982). This entry then is one that builds from real-world or natural relations.

The use of qualitative graphs as a start, suggested most clearly in the work of Krabbendam (1982), is intriguing and is the most unusual. It is clear that qualitative graphs can be taken to sophisticated levels. Using such an entry emphasizes the landmarks and conventions of graphs and may create a place holder for them in the algebraic world—an attribute Schoenfeld et al. (in press) found lacking in the education of their student, IN. The discussions that teachers must lead about the qualitative graphs are somewhat more constrained by the notational conventions, even without the numerical or scale burdens, than are comparable discussions with rule-biased tables. Thus, looking at a qualitative graph of time and growth requires some knowledge of the conventions of directionality in the Cartesian system. The entry in qualitative graphs may be either from the constructed portions of a child’s world or from an interpretive activity. Swan (1982) and others have developed sophisticated and intriguing examples for students to use as bases for discussion. Krabbendam, on the other hand, has students attempt to draw their own qualitative graphs about variables that have quantity but rarely any scale. In spite of the subject matter demands placed on the teacher, and the sense of notation that must be available, the use of qualitative graphs seems an intriguing option that has not been explored systematically. What it can do potentially is both build on and shape intuition, without the constraints of learned formalism. Further, using explorations with qualitative graphs can point up the specifics of the Cartesian system (spirals, circles, bars, etc., all appear as inventions by children) as pre-markers for the student that can facilitate later absorption of the content.

As suggested in the section on student learning, it is clear that students will carry intuitions of functions and graphing with them both from their real-world experience and from their past formal math education. Therefore, one entry point to the topic strongly endorsed by Mansfield (1985) is that of the students’ own knowledge bases. This suggests that a prerequisite teaching goal is to make public and shared the knowledge carried from math and other subjects and to work from that base (see Lampert, 1986; Leinhardt, in preparation). This has two flags: First, note that
starting this way is instructionally different from starting qualitatively (some version of this undergirds almost all recommendations); second, it requires tremendous levels of content-specific knowledge on the part of the teacher because he or she must be prepared to go in any one of several directions and to construct on the spot several curriculum scenarios. (For a discussion of why this may be problematic see Stein et al., in press). In essence, the teacher must be prepared to follow up on any of the paths discussed previously. Although at one level this may seem reasonable, it is not unlike asking a conductor to be able to finish a symphony after several members of the orchestra play their own choice of tunes. Constructing a cohesive, well-integrated, and meaningful whole may be quite difficult. The task is made easier if it is combined with a prior selection of directionality while being familiar with the implications of the various routes suggested by students.

Sequencing. The issue of entry obviously is connected to the issue of sequencing the instruction on graphs and functions. Here again, there is no empirical data on different sequences. In general, there is considerable concurrence that instruction should move from the less formal, less abstract, more global, and intuitive to the formal, notationally rigorous system. Davis (1982) emphasizes moving from one-to-one simple rules of correspondence to increasingly complex rules, including those cases where different forms of rules occur, illustrating how several forms of rules can in fact produce the same set of paired values. Bergeron and Herscovics (1982) also stress starting with intuitive views of functions, moving toward initial mathematical or quantifying experiences, and finally to abstract and formal approaches. In doing so, they tend to emphasize moving from graphs to algebraic representations.

In the traditional mathematics curriculum, algebraic functions are introduced in ninth grade, followed shortly thereafter by lessons on graphing those functions. Graphing conventions, on the other hand, often are taught earlier (often in elementary grades) and separately from the topic of functions. Janvier (1982) and Swan (1982) have elaborated a complete set of experiences that take into account both the movement from qualitative recognition or interpretation tasks through to quantitative construction tasks. It is clear from their thoughtful and careful exploration of the pedagogical issues that both authors have considered the path through the topic. Some of the material, however, may best be handled simultaneously or contrastively rather than sequentially. Roughly, the sequence they propose moves from tabulation of data, through plotting and reading of values, to sketching shapes of the graphs in preparation for matching that with functions of the same family. Interpreting various graphs then moves to formula identifications and finally curve sketching and curve fitting. The emphasis is on moving from the specific real data through the process of recognizing a function and sketching it. There are several advantages to this approach. First, it tends to draw the emphasis away from the notational issues and hence embeds the graphing experience in the task of displaying real data—a real task. Second, while it capitalizes on student interest and concrete examples, it also has a systematic plan that is directional and permits a cyclic increase in complexity.

In what can be seen as an amplification or perhaps an alternative to this sequencing, Krabbendam (1982) has urged building (in fairly formal ways) on students' intuitions about qualitative phenomena, having students construct graphs of nonquantitative but measurably covarying variables. There is an emphasis (as
with Janvier, 1982) on starting with at least one natural variable, such as time or a
time-dependent variable (e.g., growth). The approach proceeds by using qualitative
descriptions and feedback to develop “broad insights” in the student. The qualitative
consideration of two truly separate quantities can be quite difficult. In Figure 15,
from Swan (1982), skills of measuring and fitting are called into play, as is the
general concept of qualitative reasoning about graphs.

A feature of Krabbendam (1982) and Davis (1982) that is particularly interesting
is the suggestion and examples of difficult qualitative problems. That is, there is a
noticeable press to force the students to handle difficult problems qualitatively
before they confront them abstractly and quantitatively. In Davis’s Guess my Rule
examples, one rule is the absolute value; another is rounding to the nearest smaller
whole number including negative numbers. These examples provide locations for
mathematical discussions as well as advancing the particular topics of functions or
graphs. One aspect of sequencing on which there is considerable concurrence, then,
is to move back and forth between qualitative and quantitative presentations of
graphs and graphing. It is interesting that this too may help teachers to break free
of a kind of rigid, noninterpretive approach to textbook usage (Stein & Baxter,
1989). The point is that the research consistently supports a lack of relationship
between the specific, notational, analytic capabilities and the global, qualitative
ones. It can be assumed that this lack is bidirectional and that both skills are
necessary.

We continue the discussion of sequencing by making even more extensive
inferences of an author’s intent than those described earlier. Yerushalmy (1988)
conducted an empirical study of students’ knowledge of different parts of the graphs
and functions world. The specific series of questions was organized around an
implicit theory of natural acquisition sequences and thus suggests an approach to
sequencing. The sequence of questions reflects both pedagogical and subject matter
aspects of difficulty. For example, Yerushalmy notes that students can recognize,
identify, or classify families of graphs of greater difficulty and complexity than
those they can construct or otherwise generate. The specific sequence starts with

![The car C travels at a constant speed along a road in the direction of the arrows. Complete the sketch of the graph.](image)

**FIGURE 15.** Complex qualitative graphing task

*Note. From Swan, 1982, p. 159; reprinted by permission.*
identifications of graphs, moves to construction, and finally requires the learner to match graphs with families of equations. As a subset of this activity, the student also is required to identify telling features of graphs, such as shape and location in the graphical space. This stage then moves into more conventional aspects of functions and graphs by requiring determination of specific features such as slope. Finally, the learner is confronted with confounding information, such as adjustments to scale producing different pictures for the same function or similar pictures for different functions. Thus, the subtlety of the interrelationship between graphs and functions is brought out (Yerushalmy, 1988). This increasingly constrained and formal presentation is consistent with that of Swan and Janvier and seems to be an extension of it.

Explanations. Instructional explanations are a key component of mathematical instruction. In general, explanations are a part of a presentational lesson and include features such as examples and various types of representations or embodiments (see Nesher, 1989; Ohlsson, 1987). In a specific sense, explanations can be seen as ways in which meaning in mathematics is enhanced, mathematical constructs are clarified, and logical interrelationships are highlighted (Even & Ball, 1989). In a broader sense, one can consider features that good instructional explanations have in common; and one can distinguish among those explanations that are purely disciplinarily based, those that are personal and internal, and those that are intended to instruct. Expert teachers of mathematics tend to include specific elements in their explanations (Leinhardt, 1987, in press). Characteristically, these explanations include carefully selected representations and well-developed verbal expansions. The problem and the principles used in the solution are clearly flagged and marked. Specific errors are deliberately investigated rather than easily overlooked. Some sense of proof is often present, which connects the particular component to be explained to other situations that are proved in similar ways (Leinhardt, in press; Leinhardt, Putnam, Stein, & Baxter, in press).

Much of the recent research in mathematics has emphasized the benefits and necessity of a student constructing his or her own sense of meaning and understanding about a specific topic. This means, of course, that at some point the internal language and notations of the student must be linked or connected to the more conventional forms that are generally shared. Krabbendam (1982) emphasizes that all explanations should be built from the child’s own self-explanation. Krabbendam uses naturally occurring situations to establish a common base and then has the students justify their qualitative graphs. This is undoubtedly powerful. Formal testing of its efficacy, however, is still in the future. In general, researchers agree that explanations on graphs and functions should build from the students’ knowledge and include emphasis both on the confusions of language and on the meanings of scales and variables (Kerslake, 1981; Mansfield, 1985). For example, words such as line and point have particular meanings in both natural language and mathematical language; the meanings both overlap and misalign with each other.

One of the more difficult aspects of the explanations in graphing, and to a lesser extent functions, is the burden of notational conventions. It is relatively easy to explain by example the need for conventions (e.g., over, up; see Open Court’s town agreement about avenues and streets in Graph City that is a representation for graphical ordered pairs, Willoughby et al., 1981, p. 136). It is clear what happens without them or with a misunderstanding about which way they go (see Schoenfeld
Teaching the need for conventions is not the same as teaching the specific convention. The degree of explanation or lack of it for notations is in general not dealt with by most writers. In a sequence of lessons taught by Lampert (lessons observed in the fall of 1988), however, there is a clear break during which the students simply rehearse the over/up convention, having established the need for a convention (Leinhardt, in preparation). This was a conscious decision on the part of Lampert. It is interesting that other aspects of the Cartesian system, which likewise might have been treated as somewhat arbitrary conventions, were spun out in more depth to build explanations that linked well with other parts of the students' knowledge base.

Unlike the issue of notational conventions, it is quite clear from the work of Schoenfeld that extensive explanation that helps to link the graphical or Cartesian aspects to the functional aspect is critical. This part of the explanation is the bridge we referred to at the beginning of the review. Showing, for example, the connections between differing values of the ratios of the differences of x’s and y’s (m) and the differing pictorial lines must be built into any explanation of graphs and functions. Although the work of Schoenfeld and his colleagues is clearly a start, there are no microlevel descriptions of explanations and their supporting examples yet.

Examples. The selection of examples is the art of teaching mathematics. Making available for consideration by the student an example that exemplifies or challenges can anchor or critically elucidate a point. Helping teachers to identify and construct focused examples on which to build explanations is an obvious and overlooked area for intervention. Kerslake (1981) describes the use of a contrast between continuous and discrete variables on graphs to help students see the differences between them. Swan (1982) and others use nonobvious or counterintuitive graphs for qualitative discussions to make particularly salient points. Davis (1982) describes using multiple rules for Guess My Rule, some of which include steps that essentially undo previous steps. These rules then collapse to simpler ones; for example, \( y = 2x \) might be one, whereas another might be \( y = 2x + 935 - 900 - 35 \). Such examples focus the student on several aspects of rules (functions). Other rules suggested by Davis (1982), such as the absolute value or rounding to the closest smallest whole number, are counterintuitive in that the rule either does not have the common operational form (absolute value) or violates the patterns the students consider normal (raise or lower to the nearest whole number).

The power of examples as they exist in texts and are transmitted to students by teachers is evident in the work of Zaslavsky (1989). In her study, Zaslavsky showed that both correct conceptions and misconceptions presented in the example were remembered. This signals a warning often overlooked: If examples are powerful and they are incorrect, they too will be remembered as hooks for further misconceptions. In Zaslavsky’s study, all students retained the examples; it appeared that the brighter students fell back on the formal definitions, whereas the less able students reasoned from the specifics of each example remembered. (Although the data on this point are sparse, they are consistent with the finding of Chi, Bassock, Lewis, Reimann, & Glaser, in press, in their work with physics students.)

In summarizing the teaching of graphs and graphing, the emphasis must be on the intricacy of the subject and its consequent burdens on the knowledge base of teachers. The emphasis of this paper has been on the teaching of functions and graphing at an introductory level in the late elementary/middle school years and
early high school. At these grade levels, teachers often are not certified specifically within a single subject area. That does not necessarily mean that the topic is in some way too difficult to teach; rather, it seems to mean that teachers must be well supported by texts and in-service programs that help them see the interconnectedness of topics. Ideas of correspondence, mapping, rule-based relationships, and dual formalisms for single constructs are exciting and powerful mathematical notions. For teachers to communicate the specifics of these ideas, such as the specifics of proof within geometry, they must be armed with a rich base of well-annotated examples. In addition, they must understand the interconnections so that their explanations can be accurate and complete rather than fragmented and misleading (Stein et al., in press; Zaslavsky, 1989).

It is our sense that for some time to come the decisions of entry and sequencing will not be directly or solely in the hands of teachers in the way that explanations and examples are. However, researchers have only hinted at some of the more subtle issues of introduction and path, usually as proclamation rather than as empirical research. What has been well documented is that the presence of one element, such as construction of graphs from tables, does not produce capability in another area, such as interpretation of qualitative graphs. This suggests that, regardless of entry, the sequences must contain all significant elements.

Summary and Conclusion

This paper is a review of a specific part of subject matter and how it is learned and may be taught. The objective is to trace the major themes in this area, the introduction of graphing and functions in upper elementary school; but the objective is also to suggest the value of taking a subject matter-oriented perspective in reviewing teaching and learning.

We have reviewed the dual facets of construction and interpretation in graphs and functions. This two-sided activity of producing and generating, on the one hand, and interpreting and reasoning from, on the other, are present in other parts of the early mathematics curriculum (e.g., geometry), but they are most sharply present in these topics. There is also the sense that what is interpretation for graphing may well be construction for symbolic algebraic representation. In other words, the way an individual interprets a graph often involves some level of algebraic construction in order to do the interpretation, whereas at other times it involves more direct visual comparison.

We have also pointed to the intricacy of the subject from the debates surrounding appropriate types of introductory definitions of function to the subtleties of a simple notion of scale or unit. With respect to this latter point, we suggest that this is also a location for the apparent discontinuity of task that occurs between the social and physical sciences' use of graphs, graphing, and functions as contrasted to the mathematical use. The mathematical use focuses on the properties of curves and lines and their equations. The task in the social sciences is to determine the space in which to place information, or to identify the axes, scales, and data units; then to construct a graphical representation; and finally perhaps to suggest an algebraic form for the curve.

We have discussed what is known and is being researched about the nature of students' intuitions about functions and to a lesser extent graphs. We have also
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examined the types of misconceptions that researchers have identified. These, in turn, suggest the concepts that might help to anchor a student as he or she moves throughout the area. Clearly, from the work of Schoenfeld, an explicit notion of the correspondence between the simplest of graphical notions—a line and its algebraic equation—is crucial. In fact, we might say that this correspondence is the thing to be learned. Slope as a powerful linking concept is probably a good place to begin. The dual sense of directionality in the Cartesian space is another underlying concept that may be confused or lost by the early student. That is, the notion that values get bigger going up and to the right, and smaller going down and to the left is both arbitrary and fixed. But this concept tends to compete with the concept of centrality of origin and concentric increase of absolute values from the origin. (How can the origin be both the “center” and a place in a gradual drift of increasing values from lower left to upper right?) The bidirectionality of the correspondence of algebraic and graphical representation is also important, because, as the work of Zaslavsky has shown, students favor the route from algebra to graph (perhaps because that is the direction of teaching, although it is not the direction of use).

Other concepts of value include the connections between individual points, sets of points from a specific table, and all points on a line covered by a particular linear equation. Finally, a strong sense of the graphical and algebraic landscape needs to be developed in the student, including a sense of where to look for critical information. Again, the work of Schoenfeld showed that, for at least one student, landmarks in one domain—the line crossing the axes—became tangled with the landmarks in another—the single value of the slope and the y-intercept.

The teaching of graphs, graphing, and functions is understudied in comparison to other areas of early mathematical instruction (addition, fractions, decimals, algebra word problems). Actual studies of teaching at either the elementary or secondary level are quite rare and, in general, unconnected to the knowledge that a student develops. Although we have pointed to the various suggestions for entry, sequence, and the construction of explanations, we have left undescibred the role of computers and representational issues raised by them, although this is an area with some research, notably that of Kaput (1986) and Heid et al. (1988).

What do we need to further our communal understanding of this area? We need to understand what students know and the power and utility of their intuitions at different age levels and after different kinds of instruction. The power of anchoring instruction around core concepts needs to be explored. We need to have studies of instructional sequences and how they affect the learner. We also must have empirical studies of the effects of computers both as facilitators and as problems for the learner. Creative and subtle tests like those of Yerushalmy, which are attached to instructional information, would provide a tremendous amount of knowledge. The subjects for these studies should include sixth and seventh graders as well as the elementary school teacher who may soon be faced with curricula that are woven around a core understanding of the function concept.

Many authors, most notably Shulman (1986), have called for both research and reexaminations based on a subject matter perspective. The implications of doing this type of research or review of research, however, are not trivial. Educational psychologists, and perhaps policy analysts as well, must immerse themselves in a discipline and join up with subject matter experts to do initial research and to integrate strands of research. We hope that this review will have a dual value, then:
first, as a review of research on graphs, graphing, and functions in upper elementary grades, and second, as a first approximation of a subject matter-based review for educational audiences.

**Note**

1 As noted earlier, we have restricted this review for the most part to research and ideas that would be most relevant for the upper elementary through ninth grades (or through functions and graphs as they are usually presented in the first year of algebra).

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