Teacher listening: The role of knowledge of content and students

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A B S T R A C T

In this research report we consider the kinds of knowledge needed by a mathematician as she implemented an inquiry-oriented abstract algebra curriculum. Specifically, we will explore instances in which the teacher was unable to make sense of students’ mathematical struggles in the moment. After describing each episode we will examine the instructor’s efforts to listen to the students and the way that these efforts were supported or constrained by her mathematical knowledge for teaching. In particular, we will argue that in each case the instructor was ultimately constrained by her knowledge of how students were thinking about the mathematics.

The challenges associated with research-based inquiry-oriented mathematics instruction have been well documented in the literature (Ball, 1993; Cohen, 1990; Speer & Wagner, 2009; Wagner, Speer, & Rossa, 2007). At the same time, there is a growing body of research on the types of specialized knowledge required for teaching (Ball, Thames, & Phelps, 2008; Hill et al., 2008; Shulman, 1986). However, much less is known about how these types of knowledge support teachers as they engage in inquiry-oriented instruction (Speer & Wagner, 2009).

One notable exception that begins to address this gap in the research base is a study done by Speer and Wagner (2009) in which they reported on the difficulties of providing analytical scaffolding during whole class discussions; where analytic scaffolding is used to “support progress toward the mathematical goals for the discussion” (p. 493). To do this Speer and Wagner used a composite framework drawing on components of Ball et al.’s (2008) mathematical knowledge for teaching construct and Shulman’s (1986) idea of pedagogical content knowledge. They argued that one specific domain of mathematical knowledge for teaching, specialized content knowledge, is needed to support teachers in coming to understand the ideas expressed by students during a discussion, while pedagogical content knowledge is needed to support teachers in determining whether these ideas have the potential to contribute to the mathematical goals of the discussion (and contribute to the students’ learning).

We aim to build off the work of Speer and Wagner (2009) by further explicating the role of pedagogical content knowledge in supporting inquiry oriented instruction. Speer and Wagner focused on classroom discussions in which students contributed potential solutions. The mathematician who was leading the discussions struggled to make sense of these solutions and to determine whether they had the potential to move the mathematical conversation forward. In their analyses, Speer and Wagner found that it was the special mathematical work of making sense of the students’ contributions that gave the instructor (a research mathematician) the most difficulty.

The teaching episodes we will describe are somewhat different. We will be looking at instances in which students expressed difficulties with the mathematics, whereas Speer and Wagner looked at whole-class discussions in which students shared solutions. As a result, our analyses contribute new insights into the role of teacher knowledge in supporting inquiry-oriented instruction. Additionally, we note that in their analyses Speer and Wagner (2009) used Shulman’s (1986) general

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pedagogical content knowledge construct. Ball et al. (2008) refined this construct identifying three sub-domains (knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum). Our research further explicates the role of one of these sub-domains, knowledge of content and students, in supporting teachers in the work of identifying and understanding students’ mathematical struggles.

1. Theoretical perspective

In an effort to identify specific ways in which mathematical knowledge for teaching (Ball et al., 2008) impacted the implementation of an inquiry-oriented curriculum, we decided to focus on the knowledge needed to support mathematicians as they listened to their students. Davis (1997) discussed three different types of listening that a teacher can engage in: hermeneutic, interpretative, and evaluative.

When a teacher engages in hermeneutic listening, the teacher becomes “a participant in the exploration” (Davis, 1997, p. 369). As such, the traditional roles of teacher and student begin to shift, and the class, rather than the teacher, becomes the mathematical authority. Additionally, when the teacher engages in this type of listening, the mode of instruction shifts to be “more a matter of flexible response to ever-changing circumstances than of unyielding progress toward imposed goals” (p. 369). This concept of hermeneutic listening was further built upon by Rasmussen (in Yackel, Stephan, Rasmussen, & Underwood, 2003) with what he called generative listening. By engaging in generative listening, the teacher is able to revise the lesson trajectory in response to student contributions. Because of this change in the trajectory of the lesson, generative listening has the ability to “generate or transform one’s own mathematical understanding and it can generate a new space of instructional activities” (p. 117). For example, by attending to a student’s novel conjecture, an instructor may learn some new mathematics and discover an opportunity to engage the classroom community in an unexpected and productive line of inquiry.

We argue that generative listening requires the teacher to listen to students with the intent of making sense of their thinking. Davis (1997) refers to this type of listening as interpretative listening, which he characterizes as listening to students with the aim of “making sense of the sense they are making” (p. 365). Both generative and interpretative listening are contrasted by evaluative listening (Davis, 1997). Evaluative listening is listening with the intent of “trying to simply assess the correctness of the student responses” (p. 365), where the students’ responses play no role in determining the trajectory of the lesson.

When analyzing how mathematicians implement inquiry-oriented curriculum, especially curriculum that is heavily influenced by the Realistic Mathematics Education notion of guided reinvention (Freudenthal, 1991), we argue that it is of great importance to look at how teachers listen to their students.

The intent of guided reinvention is that students come to view the mathematics as their own creation. One way for this to occur is for the classroom participants (teachers and students) to lay down a mathematical path as they go, rather than follow a well-trodden trajectory. (Yackel et al., 2003, p. 117)

In order for the classroom participants to be able to create the “mathematical path as they go,” student contributions need to shape the trajectory of the lessons. This requires the teacher to listen to the students with the intention of understanding the mathematics being expressed by the students (interpretative listening), and then, when appropriate, leverage the student contributions to inform the trajectory of the lesson (generative listening).

As reported by Speer and Wagner (2009), a mathematician’s ability to understand and leverage students’ mathematical reasoning in the moment may be heavily reliant on specialized content knowledge (Ball et al., 2008) and pedagogical content knowledge (Shulman, 1986). For instance, Speer and Wagner concluded that, in order to provide analytic scaffolding for whole class discussions, it is necessary to draw on pedagogical content knowledge regarding “typical ways of student thinking” (p. 557). This specific type of pedagogical content knowledge was further elaborated in Ball et al.’s (2008) framework of mathematical knowledge of teaching, where knowledge of content and students was defined as “knowledge that combines knowing about students and knowing about mathematics” (p. 401). In addition to knowledge of the mathematical content, “teachers must anticipate what students are likely to think and what they will find confusing” (p. 401).

A primary goal of our research is to build on Speer and Wagner’s (2009) observation that pedagogical content knowledge is an important factor for supporting mathematicians as they implement inquiry-oriented curriculum. Specifically, our aim is to further explicate the role of pedagogical content knowledge in listening to students. As we will illustrate, we identify knowledge of content and students as being a particularly important category of pedagogical content knowledge when it comes to the instructor’s ability to listen productively to students. Ball et al. (2008) alluded to this relationship when they proposed that knowledge of content and students is drawn upon to “hear and interpret students’ emerging and incomplete thinking as expressed in the ways that pupils use language.” (p. 401). Additionally, our findings serve to further flesh out the knowledge of content and students category, as our analyses reveal interesting differences among the types of knowledge of content and students that these kinds of listening require.

2. Background: a brief discussion of the curriculum

As part of a curriculum development and research project in abstract algebra, we have been investigating teachers’ implementation of an inquiry-oriented abstract algebra curriculum (Larsen, 2009; Larsen, Johnson, Rutherford, & Bartlo,
2009; Larsen, Johnson, & Scholl, 2011). The Teaching Abstract Algebra for Understanding (TAAFU) curriculum is a research based, inquiry-oriented abstract algebra curriculum that actively engages students in developing the fundamental concepts of group theory. The TAAFU curriculum was primarily designed to be used in an upper-division, undergraduate abstract algebra course and is composed of three primary units: groups/subgroups, isomorphisms, and quotient groups. Each unit begins with a reinvention phase in which students develop concepts based on their intuition, informal strategies, and prior knowledge. This is followed by a deductive phase in which students prove important results based on their formal definitions and previously established results. For our purposes, it will suffice to briefly describe relevant aspects of the groups/subgroups unit and the quotient groups unit.

The reinvention of the group concept begins with an investigation of the symmetries of an equilateral triangle (Larsen, 2009; Larsen & Zandieh, 2007). Students develop symbols for the six symmetries and develop a calculus for computing combinations of symmetries. The rules the students use to compute combinations include the group axioms along with relations specific to this group. Based primarily on their work within the symmetry context, the students construct a formal definition of a group. Later, the students work with this definition to create a characterization theorem for subgroups – they are asked to identify a minimal set of conditions necessary to ensure that a subset of a group is a subgroup.

The quotient group unit (Larsen et al., 2009) launches in the context of the group of symmetries of a square, $D_8$. Students begin by identifying partitions that behave like the sets of even and odd integers. Generalizing this idea, the students attempt to partition $D_8$ into four subsets that form a group (with respect to set “multiplication”). As this investigation unfolds, students produce a number of conjectures as to what conditions are necessary (and/or sufficient) for a partition of a group to form a group. Formally established, these conditions comprise the definitions and theorems of basic quotient group theory.

Our results will be presented in the form of three classroom episodes that illustrate the importance of knowledge of content and students in supporting teachers’ ability to listen to their students. Two of these episodes took place as the class worked through the groups/subgroups unit, while the other took place during the quotient group unit. More specific details regarding the instructional activities will be provided as necessary in Section 4.

3. Methods

3.1. Data corpus

To understand the ways that instructors might engage with the TAAFU curriculum, including the challenges they may face during implementation, we collected data from the classrooms of three mathematicians over a period of two years. Over these two years, three rounds of data collection took place – encompassing four implementations of the curriculum. Each of these implementations took place at the same urban, comprehensive university in the Pacific Northwest. Typical students in this course were junior and senior math majors, with a significant percentage of students planning on becoming high school math teachers. Class sizes ranged from 22 to 28 students, and each of the three mathematicians volunteered to use the TAAFU curriculum.

The majority of this report will focus on classroom videotape data from our last round of data collection. During this round we videotaped each class session as Dr. Bond implemented the TAAFU curriculum for the first time. Dr. Bond received a PhD in Mathematics (specializing in topology). This was Dr. Bond’s first time implementing the TAAFU curriculum, however she had taught abstract algebra courses before. Additionally, Dr. Bond had previously worked on a K-12 professional development project during which she received training on reform-oriented instructional practices (including methods for promoting discourse, problem solving, invention, and inquiry). Further, as part of three summer institutes associated with the professional development project, Dr. Bond taught a discrete mathematics course to in-service teachers while modeling these practices.

During Dr. Bond’s course we videotaped all regular class sessions and held regular videotaped debriefings (twelve total) throughout the 8-week summer term. Additionally, the first author observed and took notes during each class session.

3.2. Data analysis

Consistent with Lesh and Lehrer’s (2000) iterative video analysis technique, we made four passes through the data. Initially, the research team looked at classroom video data from all three of the mathematicians’ classes. The goal of this first pass of data analysis was to identify episodes in which students made contributions, and to record both the student contributions and how the mathematician responded to the contributions. Then, in our second pass of analysis, we coded these instances both for the teacher responses and the student contributions. Preliminary codes for teacher responses included “teacher provided counterexample” and “teacher explores conjecture,” and preliminary codes for student contributions included “useful” and “unexpected”. A contribution would have been considered unexpected if the research team did not anticipate a task evoking such a contribution. A contribution would be considered useful if the research team believed a contribution could be leveraged to move the mathematical agenda forward.

We found that all three mathematicians were generally successful at implementing the curriculum, sometimes doing an exceptional job making sense of, and leveraging, student ideas. However, our analysis revealed a number of instances in which

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1 All teacher and student names are pseudonyms.
students expressed confusion or difficulties understanding the mathematics, and their teachers were not able to make sense of them. We found these episodes to be particularly useful in exploring the role of the teachers’ knowledge in supporting their efforts to make sense of student reasoning. In particular we found an interesting variety of such episodes in Dr. Bond’s class. Thus, we selected Dr. Bond’s class for further rounds of data analysis, focusing on instances in which students in her class expressed difficulties. To analyse these instances we made a third pass in which we reanalysed episodes from Dr. Bond’s class using a listening framework (Davis, 1997; Yackel et al., 2003) to categorize Dr. Bond’s interactions as evaluative, interpretative, or generative. Finally, we analysed these episodes again, attempting to account for the kinds of listening we were observing by considering the kinds of knowledge described in Ball et al.’s (2008) mathematical knowledge for teaching framework.

4. Results

In this section we present three classroom episodes that illustrate the role of knowledge of content and students in supporting instructors’ ability to listen to their students. These three episodes were chosen because in each episode we can see students expressing, sometimes quite articulately, confusion with the mathematics, yet in each case Dr. Bond was unable to make sense of the students’ mathematical struggles. In the first episode the instructor struggled to make sense of a concern raised by a student. She did not seem to understand how the student was thinking about the operation of combining two symmetries, and how this way of thinking would impact how he thought about working with equations involving symmetries. In the second episode students repeatedly made the same conjecture about quotient groups after the instructor had presented a counterexample. She did not seem to notice that the counterexample was not convincing to the students, nor did she realize that it was likely that some of her students did not see this counterexample as being related to quotient groups. Finally, in the last episode, Dr. Bond repeatedly (and unconvincingly) responded to a question with a cursory argument. Each time she seemed to be unaware of how difficult it would be for students to make sense of this argument on the fly.

After describing each episode, we will discuss the instructor’s efforts to listen to the students and the way that these efforts were supported and constrained by her mathematical knowledge for teaching. Our findings differ from those of Speer and Wagner (2009) because the instructional episodes we are considering are somewhat different in nature. The episodes analysed by Speer and Wagner featured a mathematician’s struggles to make sense of students’ proposed solutions. The episodes described here are driven by students’ doubts and uncertainty. Likely for this reason, we found that the instructor did not struggle so much with the mathematical work of making sense of the students’ contribution, but rather with the work of making sense of the cognitive complexity underlying the students’ struggles. In particular, we will argue that in each case the instructor was ultimately constrained by her knowledge of content and students.

4.1. Episode I: left vs. right multiplication

The TAAFU groups/subgroups unit begins with an investigation of the symmetries of an equilateral triangle. The students express each symmetry in terms of a flip across a vertical axis (denoted \( F \)) and a \( 120\degree \) clockwise rotation (denoted \( R \)). The students then develop rules for calculating combinations of symmetries. These rules include relations specific to the symmetries of an equilateral triangle as well as some of the group axioms. For example, a student might use the associative property, the inverse property, the identity property, and the fact that \( F \) is its own inverse to determine that \( F(FR) = R \). A key step in the reinvention process involves the pruning of the students’ list of rules (Larsen, 2009). In particular, students usually generate several equivalent versions of the dihedral relation, which express the quasi commutativity of the group of symmetries of an equilateral triangle. The students prune their list of rules by proving that some of these rules follow from other rules and hence can be eliminated from the basic set of rules.

In our first episode the class was working on just such a task, showing that \( FR = R^2F \) follows from other rules (i.e. the group axioms and another version of the dihedral relation, in this case \( FRFR = I \)). One method the class had used for showing that a particular rule was a consequence of the others was to start with a given statement (such as \( FRFR = I \) or \( I = I \)) and, by manipulating that given statement, arrive at the rule they were trying to eliminate. While such an argument would have shown that \( FR = R^2F \) was a consequence of the other rules, one group actually presented this argument in reverse, by starting with the conclusion, \( FR = R^2F \), and arriving at a given statement, \( FRFR = I \) (see Fig. 1). A similar error had been made previously in class, and in that instance the class worked to correct the proof by reversing the steps. Likewise, when this proof was

\[
\begin{align*}
FR &= R^2F \\
FRFR &= FRFR \\
FRFR &= F(RR^2)F \\
FRFR &= FIF \\
FRFR &= FF \\
FRFR &= I
\end{align*}
\]

Fig. 1. Group’s proof under discussion.
presented and found to be erroneous, Dr. Bond asked the students if it would be possible to reverse the steps to arrive at a valid proof that $FR = R^2F$. However, because of the way this proof was written, reversing the first step ($FR = R^2F$ implies that $FRFR = FR^{2}F$) required using the cancelation law, which had not yet been established.

In an effort to draw the students' attention to this cancelation step, Dr. Bond first asked the students, "Are we allowed to multiply\(^2\) both sides? So that's a question when we are reading down, are we allowed to multiply by the same thing on both sides?" Initially the students agreed that, yes, as long as you multiply by the same thing on each side and do not commute elements, this step was fine. Dr. Bond then directed the class to consider that step if the proof was reversed. "At that same step if we are reading backwards, what happens there? Are we allowed to do that?" However, following this question a student, Adam, expressed concern about the multiplication step rather than the cancelation step that Dr. Bond had in mind.

Adam: Well, I wanted to note that everyone agreed that if you're at two equivalent equations, like if the left side equals the right side, then you are saying that both of the shapes are in the exact same position. So obviously if you do the same operation to the end of them, without question that has to work. They are in the same spot and you are doing the same thing to both sides. But this is different from that.

Dr. Bond: So reading down you feel comfortable with . . .

Adam: No, I'm saying that if on the left side it's some string of things and on the right side it's some string of things that we are saying are equal, and if we add two equivalent operations to the end of both sides, then it's gonna not change anything, so you should be able to do that. But adding two things to the beginning that are equivalent seems a little bit more confusing.

So, here we see that Adam was not focused on the use of the cancelation law if the proof were to be reversed, as Dr. Bond was intending. Instead Adam questioned the validity of the step in the original proof where both $FR$ and $R^2F$ were left-multiplied by $FR$. While it made sense to him that right multiplication preserves equivalence, he was unsure as to why left multiplication preserves equivalence. Dr. Bond struggled to make sense of this distinction between left and right multiplication, even after a second student, Mark, reiterated Adam's sentiments.

Dr. Bond: I don't understand the difference between your beginning and your end.

Mark: He's saying we are doing the $FR$ operation before the operations we already had.

Dr. Bond: So opposed to putting the $FR$ on this side?

Mark: Yes, if we had the triangle in the equivalent state, then whatever we did from there would clearly not change anything.

Adam: Exactly

Dr. Bond: If you start with something that's balanced . . . oh if you start with $FR = FR$ and then add another piece?

Mark: No, doing $FR$ before equivalent operations is different in his mind than after.

Dr. Bond: I'm not sure if I'm 100% clear on that but that's okay.

We can see Adam expressed that, for him, left multiplication and right multiplication were not conceptually the same. Thus, while he saw no need for further justification of right multiplication, he did see need for further justification of left multiplication. His concerns were then restated by another student, Mark. However, even given both students' attempts to explain the issue, Dr. Bond struggled to make sense of the conceptual difference the students were seeing between left and right multiplication. Thus, we would consider this an instance of unsuccessful interpretive listening.

Near the end of this exchange Dr. Bond said, "oh if you start with $FR = FR$ and then add another piece?" This may have been an acknowledgment that one may re-conceive the left multiplication by $FR$ as starting with $FR = FR$ and then right multiplying by the equivalent symmetries (in this case $FR$ and $R^2F$). Such a re-conception of left multiplication would both address Adam's concerns, and draw on the justification of right multiplication described by Mark and Adam. Unfortunately, Dr. Bond's observation by itself was insufficient to address Adam's concern and she did not follow up on her thought – not surprising given that she was unable to make sense of the students' explanations that left and right multiplication are different.

Dr. Bond's inability to make sense of Adam's statement constrained her ability to address his difficulties. Had she understood the distinction Adam was describing, she could have explicitly posed to the class the question of whether it is valid to multiply both sides of a symmetry equivalence on the left by the same symmetry. It is likely that the students could have come to the realization that a statement of the form $a \rightarrow b \rightarrow ca = cb$ could be reconceived in terms of multiplying the two sides of the equivalence $c = c$ on the right by the two (different but equivalent) elements $a$ and $b$. If not, Dr. Bond could have proposed this approach for their consideration.

\(^2\) Here Dr. Bond used the convention of referring to a group operation as multiplication. We will use this same convention throughout the paper.
Why was Dr. Bond unable to make sense of the distinction Adam was trying (quite articulately) to make? Some insight to this question comes from Dr. Bond’s debriefing interview following this class. In this debriefing session the interviewer (and first author) once again rephrased Adam’s statement, this time making direct reference to a time based, sequential conceptualization of multiplying symmetries.

Estrella: He was saying if I know two symmetries are the same, and I put a symmetry after them, so if I know a and b are equal, afterwards he could see why ac would equal bc. Cause if I move them, and then do something new, he could see why the end result would be the same. But he didn’t know why a = b would mean that ca = cb, because when you come before it’s like your going back in time.

Dr. Bond: Oh, he was thinking back in time. I get it. Yeah I couldn’t quite… I didn’t see the difference. To me there is no difference in being before or after. But he was thinking of it in a time sequence.

Provided with this extra piece of information, that Adam was thinking about multiplying symmetries as a left to right sequential procedure, Dr. Bond was able to make sense of his concern. Therefore, we argue that she was constrained by a limitation in her knowledge of content and students. Specifically, we contend that Dr. Bond was constrained by a lack of knowledge about how her students might have been thinking about the operation of composing symmetries. Students (from middle school through abstract algebra) sometimes view binary operations in terms of performing a left to right, sequential procedure (Brown, DeVries, Dubinsky, & Thomas, 1997; Kieran, 1979; Larsen, 2010). Further, the students’ experiences in the TAAFU curriculum of physically manipulating triangles in order to perform combinations of symmetries would serve to keep this procedural view at the forefront of their thinking. This view of operation is reflected in Adam’s description of what right multiplication does to the position of the triangle: “You are saying that both of the shapes are in the exact same position. So obviously if you do the same operation to the end of them, without question that has to work”.

Armed with this specific knowledge of content and students, which we would not expect someone to have unless they were familiar with the research literature, Dr. Bond would have been in a much better position to make sense of Adam’s concerns about left multiplication, as we saw her successfully do during her debriefing interview. Thus, we see this episode as evidence of how knowledge of content and students can play a critical role in supporting instructors’ interpretive listening.

4.2. Episode 2: quotient group conjecture

A foundational task in the TAAFU quotient group unit (Larsen et al., 2009) requires students to partition the group of symmetries of a square, $D_8$, into subsets that form a group, called a quotient group, under the operation of set multiplication $(A \times B = \{a \in A, b \in B\})$. As part of this process, the students conjecture and prove that the identity subset of a quotient group must be a subgroup of the original group. Immediately following this task, the students are asked to determine how a group needs be partitioned, after choosing a subgroup, in order to form a quotient group. The student conjecture featured in this episode was discussed at different times as the students were working on these two tasks.

It is important to note that, at this point in the curriculum, the students did not have a formal definition of quotient group. Rather, the class had a working definition that said, when one breaks up a group to make a new group consisting of subsets, that group is called a quotient group. This working definition matches what the students had done in class, having partitioned groups ($D_6$ and $D_3$) into subsets such that these subsets, under set multiplication, satisfy the group axioms.

The relevant episode commenced as the class was trying to show that the inverse element of each element in the identity subset must also be a member of the identity subset (part of the proof that this subset must be a subgroup). A student, Matthew, observed that, “they’re all self-inverses...all the elements” (where he was referring to elements of the quotient groups). In fact, it is true that every element of every proper quotient group of $D_8$ is a self-inverse. Dr. Bond responded by saying that, “[while] that is a true observation for every single one of ours, that is one that is not necessarily going to be true in general. But we don’t have an example of that, so now I just know more than you.” It is at this point that a student, Keith, volunteered, “The homework, the homework with the modulo?”

Keith was referring to a homework assignment that the class was currently working on (Fig. 2), in which $Z_n$ was introduced through the division algorithm and the operation was defined in terms of representatives. The students were asked to show that this operation is well-defined, and that $Z_n$, under this operation, forms a group. Note that, while it is true that $Z_n$ is an example of a quotient group, both the method for constructing the elements and the description of the operation provided in the assignment were significantly different from the what the students had experienced thus far in their work with quotient groups. Further, these groups were not identified as examples of quotient groups.

Dr. Bond responded to Keith’s suggestion by pointing out that $Z_n$ does meet the classroom community’s working definition of quotient group.

Dr. Bond: It is, in fact, a quotient group by the definition we put up at the beginning of class. When we break up a group, like the integers, to make a new group of subsets, this group is called a quotient group. And when you look at that group there are definitely examples in which elements aren’t their own inverses.

Dr. Bond then pointed out that, in $Z_3$, [1] is not its own inverse.
1. The division algorithm states that if $a$ and $b$ are integers with $b > 0$, then there exists unique integers $q$ and $r$ such that $a = bq + r$ where $0 \leq r < b$. We can use the division algorithm to define the integers modulo $n$, $\mathbb{Z}_n$.

$$\mathbb{Z}_n = \{[0], [1], \ldots, [n-1]\}$$

where $[k]$ represents the set of integers that have remainder $k$ when divided by $n$.

Example: $\mathbb{Z}_4 = \{[0], [1], [2], [3]\}$ and

$$[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$$

$$[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$$

$$[2] = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$$

$$[3] = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$$

Although it is standard to use the integers $0, \ldots, n - 1$ to represent the elements of $\mathbb{Z}_n$, any element of the set can be used as a representative. For example, in $\mathbb{Z}_4$, $[1] = [5] = [-3]$ and so on.

We can define an operation on $\mathbb{Z}_n$ by $[a] + [b] = [a + b]$ where $a$ and $b$ can be any representative.

(a) Show that if $[a_1] = [a_2]$ then $a_1 - a_2 = kn$ for some $k \in \mathbb{Z}$. Note, that this relationship is an equivalence relation. The equivalence classes under this relation are the elements of $\mathbb{Z}_n$.

(b) Show that if $[a_1] = [a_2]$ and $[b_1] = [b_2]$ then $[a_1 + b_1] = [a_2 + b_2]$. Note, this proves that the operation defined above is well-defined.

(c) Prove that $\mathbb{Z}_n$ with the operation defined above is a group.

Fig. 2. Homework introducing $\mathbb{Z}_n$.

The next day, while the students worked to determine how the elements of a group needed to be partitioned given a subgroup of order two, another student, Adam, conjectured that all of the elements of a quotient group are self-inverses and Dr. Bond again used $\mathbb{Z}_n$ as a counter-example.

Adam: Well, the one thing I thought of, and this only holds for sure if there's four, like if there's going to be four subsets... say you have your four brackets, your four subsets are like $I$, $A$, $B$, and $C$, where one of them is the identity. Then, you know that $A$ has to be its own inverse, $B$ has to be its own inverse, and $C$ has to be its own inverse.

Dr. Bond: No

Adam: That's not true? For four, yeah it is. How could it not be?

Dr. Bond: Ah, the rotations of the square... I'm giving you a group that has four elements where not every element is its own inverse.

Adam: No, I'm only saying the subset $A$ has to be its own inverse. That's not true? It's true in this case. It's not always true?

Dr. Bond: Well, we talked about that a little yesterday, remember? Someone brought up the example of the $\mathbb{Z}_n$. Those are quotient groups. The elements are subsets of $\mathbb{Z}$, right? And not every element is its own inverse. In particular if you look at $\mathbb{Z}_4$, the element $[1]$, which is the subset that has all the numbers that have remainder 1 when you divide by 4 so it's 1, 5, 9, 13. That element, when you add it to itself, you do not get back to the identity element, which is multiples of 4. Because when I add it to itself I get things like 1 plus 1 is 2, and 1 plus 5 is 6, and 1 plus 9 is 10. I get the numbers that have remainder 2.

Adam: Weird, I messed up somewhere. I tried to prove it in general. I wasn't just saying that.

We can see from Adam's last comment that he was still unsettled. Dr. Bond's example raised doubt but did not seem to resolve the issue for him. Further, these two refutations of this conjecture by appealing to the $\mathbb{Z}_n$ counterexamples did not prevent other students from proposing it. Shortly after this exchange, a couple of students working in different groups recognized that in each of the examples of quotient groups the class had constructed, any time you multiplied two elements of the same subset the result was a member of the identity subset.

Mark: I just observed that the one that we worked, in the beginning, the combination of the two elements always gives you an element of the identity set. I have no idea if that actually leads somewhere.
To further explore this idea, Dr. Bond again led the students to $Z_4$ as a way to check if this observation could be translated into a viable conjecture. To do this, Dr. Bond listed the elements in $Z_4$, and drew the students’ attention to one of its elements, [1]. She then asked the class what would happen if two of the elements within the set [1] were combined. If Mark’s observation was true in this case, then the combination of any two elements in the set [1] should result in an element from the set [0], where [0] is the identity of $Z_4$. However, when Dr. Bond posed this question, “if I take two of these guys (points to the list of elements in [1]) and I combine them, do I get an element of the identity?”, the following unexpected exchange occurred.

Keith: The identity subgroup would be zero and two and the other one would be one and three.

Mark: But zero is the identity subgroup.

Dr. Bond: Zero is the identity subgroup. That’s the one when you add it to anything you get it back.

It is of note that Keith was the student who initially suggested that the $Z_n$ groups would be relevant to the discussion. However, we can see from his statement above that he is not thinking of $Z_4$ as a quotient group itself, but as a group that could be partitioned to form a quotient group - he correctly describes a partition of $Z_4$ into two cosets, Mark, on the other hand, does seem to be thinking of $Z_4$ as a quotient group itself, stating that [0] is the identity subgroup. Dr. Bond does not seem to recognize the incongruity (or significance) of Keith’s comments. In fact she never seems to consider the possibility that some of the students were not seeing $Z_4$ as an example of a quotient group. When this example was first suggested by a student, and again on the second day when she recalled the example herself, Dr. Bond simply pointed out that it was an example of a quotient group because its elements are subsets of $Z$. She gave no time for the students to process this claim but instead simply moved on to say that it is an example in which not all elements are self-inverses.

From here, Dr. Bond worked through an example of combining two elements in [1], showing that indeed [1] is not its own inverse. During this work Dr. Bond had an “ah-ha” moment about why this conjecture does not hold in $Z_4$, “well-duh because one plus one is two. And that’s what’s going on there. Those elements [for which the conjecture was holding] are all order two, and so when you multiply them with themselves you are supposed to get the identity back.” Here we see Dr. Bond realize that Mark’s observation – combining two elements in the same subset always results in an element in the identity subset – is equivalent to saying that the subset must be its own inverse.

This “ah-ha” moment Dr. Bond experienced during this episode is worth commenting on. By engaging with her students’ ideas, Dr. Bond learned a new way of thinking about what it means for an element of a quotient group to have order two. While not a major mathematical achievement, this did enable Dr. Bond to connect two student conjectures that she had previously seen as different and to share this connection with the class.

Further, this “ah-ha” moment may give insight into the nature of specialized content knowledge (Ball et al., 2008) in the case of mathematicians. This is the course of mathematical research it is common for mathematicians to have to make sense of new and different ways to think about mathematics with which they are already familiar. This is often the way that new progress is made on open problems. Likewise, student contributions are typically presented in unfamiliar ways, especially when enacting a curriculum for the first time. This may suggest some similarity between the work done by professional mathematicians in a research setting and the work involved in understanding and evaluating the mathematics that students do in the classroom – the kind of mathematical activity Ball et al. connect to specialized content knowledge. It may be the case that research mathematicians possess the mathematical tools needed for developing specialized content knowledge for teaching, but need opportunities to engage with students’ mathematical thinking in order to develop this kind of knowledge.

Focusing on how Dr. Bond engaged with students as they tried to develop a coset formation algorithm, we see Dr. Bond taking an unexpected student conjecture and exploring that conjecture with the class. Because Dr. Bond allowed the students’ contribution to influence the trajectory of the lesson, we see this as an example of generative listening. The students were able to test their conjecture, consider properties that were equivalent to their conjecture (in this case we see that the combination of any two elements in a subset resulting in a member of the identity subset is equivalent to the subset being a self-inverse), and explore the quotient group concept in a setting other that the dihedral groups. Unfortunately, because some students appeared not to have seen $Z_4$ as a quotient group, this teacher-directed exploration may have had a limited impact. Indeed, it is in regards to how Dr. Bond responded to student comments about $Z_4$ within this example of generative listening that we find an example of evaluative listening.

When Dr. Bond asked the students what it would look like to test the conjecture within the $Z_4$ example, Keith responded by trying to partition $Z_4$ into a quotient group, as evidence by his statement, “the identity subgroup would be zero and two and the other one would be one and three.” In response to Keith’s contribution, Dr. Bond reiterated that [1] was not its own inverse (the same counterexample she had used to address Adam’s statement that every element in a quotient group was its own inverse) and then, after Mark gave a response that matched her interpretation, Dr. Bond went on to test the original student conjecture using an example that she knew was a quotient group, $Z_4$. Given Dr. Bond’s reaction to Keith’s contribution, is it possible that she either: (1) did not listen to what Keith said; (2) evaluated it as incorrect or incompatible and thus chose to ignore it; or, (3) was unable or unwilling to try to interpret his statement. For this reason, we consider this to be an example of evaluative listening.

It is likely that Dr. Bond chose to test this conjecture (and refute Adam’s claim that every element in a quotient group is a self-inverse) with $Z_4$ because the students had been exposed to this type of group on their homework and because a student had brought it up in the previous class. Therefore, Dr. Bond probably believed that she had selected a quotient group example that was familiar to the students and that would allow them to test their conjecture in a context they understood. Given this
rationale, the selection of $Z_4$ to test this conjecture reflects Dr. Bond’s knowledge of content and teaching, a domain of Ball et al.’s (2008) pedagogical content knowledge that refers to mathematical knowledge specifically related to instruction.

However, though $Z_4$ is an example of a quotient group, and the homework assignment shows the elements of $Z_n$ expressed as sets, it is likely that the students did not see $Z_4$ as a quotient group, as evidenced by Adam’s and Keith’s comments. This disconnect seems reasonable because the students had not yet fully developed the quotient group concept. Prior to this homework, the only examples of quotient groups the students had to draw on were formed by partitioning the dihedral groups $D_3$ and $D_4$ into subsets that acted like elements of a group under set multiplication. The homework given asked students to show that $Z_n$ was a group, not that $Z$ could be partitioned in a way that the subsets acted as a group (a quotient group).

The exploration of the students’ conjecture was inconclusive in spite of the fact that Dr. Bond exhibited the necessary specialized content knowledge (she was able to make sense of the students’ conjecture and later connect it to an equivalent version she was familiar with) and exhibited good knowledge of content and teaching in selecting what would typically be an appropriate counterexample to the conjecture. However, she appeared to be unaware that some students, so early in the development of the quotient group concept (and with so little experience working with $Z_n$), were unlikely to be thinking of $Z_4$ as a quotient group at this point. So for her students, in this situation, the counterexample was likely not appropriate.

We see this as a situation in which Dr. Bond was constrained by her knowledge of content and students. During the debriefing session that followed this class, Dr. Bond did acknowledge that the time spent in class working with $Z_n$ was useful because these groups are “a lot for them to process.” Thus it appears that she was aware that the students struggled to make sense of the example she was using to respond to their conjecture. However, she did not seem to realize that the students were unlikely to see the $Z_4$ groups as a counterexample to their conjecture (as they were unlikely to see the $Z_n$ groups as examples of quotient groups).

4.3. Episode 3: identity and inverses in a subgroup

In this final episode, the students were working to identify a minimal set of conditions needed for a subset of a group to be a subgroup. In order to succeed in this task, the students needed to realize that the identity of a subgroup must be the same as the identity of the group (and, similarly, the inverse of any element in the subgroup is the same as its inverse in the larger group). In this section we will explore events in Dr. Bond’s classroom as she engaged with the students’ struggles/concerns regarding this fact.

Dr. Bond introduced the idea of subgroup by proving that $5Z$ under addition is a group in which every element is a member of a larger group, $Z$, under addition. It is during this discussion that students began to question how they knew that the identity of $Z$, 0, acts like the identity in $5Z$. After Dr. Bond had completed the proof that $5Z$ was indeed a group, one student, Molly, stated “I’m just talking about in the identity . . . it seems like it, we would need to show that would be true”. Dr. Bond responds by stating that, in that portion of the proof, she was identifying what the identity of the group is for the reader. She goes on to say “I know it has an identity and because that’s an example we’ve worked with frequently. It’s fine to just say this is the identity of the group”. In this instance, Dr. Bond does not seem to consider that Molly may have been questioning how they know that 0 acts as the identity of $5Z$.

Following this interaction, the class then began working in small groups to develop conjectures about the minimal list of criteria needed to ensure that a subset of a group is a subgroup. During small group discussion one group conjectured that, in order to know that a subset of a group is a subgroup, it is enough to check that the inverse of each element in the subset is also in the subset, and that the subset is closed under the group operation. They argued that, because any element combined with its inverse produces the identity, in order to prove that a subset of a group is a subgroup, you only need to make sure the subset is closed and that the inverse of each element in the subset is a member of the subset. Even though this is not the standard subgroup theorem, it is a common, and valid, conjecture that students typically develop while working through the TAAFU curriculum (Larsen & Zandieh, 2007). However, also during small group discussion, some students were questioning how one would determine the inverses of the subset elements and the identity element of the subset. For example, Matthew told his group, “Well, we still have to check if the identity is the identity of the set . . . I’m not a hundred percent positive”.

To initiate whole class discussion Dr. Bond repeated the conjecture she heard from the students and pointed out that this conjecture, and line of reasoning, depends on what the identity and inverse elements of a subgroup are. In response, a student, Sarah, provided a justification as to why the identity element of a subgroup must be the same as the identity element of the group.

Dr. Bond: What I thought I heard John say was, if we know it’s closed and we know we have inverses then we’ll get the identity. Because when you take an element and you add it to its inverse, we know. Now that argument is a nice one, but it really, you have to think about it. We are really in a different setting when we are in the subgroup setting right? Because we are all like, “well do we know that inverses exists?” So, do we know that inverses exist? Do we know that this element $a$ in our subset has an inverse? Do we know that the identity exists? What’s our setting here? I think the answer is yes. Why do we know that there is an identity? Why does $a$ have an inverse?

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1 Technically, of course, one must also assume the subset is non-empty.
Sarah: I was thinking that the identity, that the identity of the subgroup will always be the same as the identity of the group . . . So we don’t have to check the identity because it has to be the same.

Dr. Bond: Yeah, but you do have to check that it is in the subgroup. So there is still something to check for the subgroup but it is different. We are not checking that the identity exists. . . . And just the same way that our subgroup is going to have the same identity, our elements are going to have the same inverses. They have a unique inverse in the group and that’s the inverse it has to have in your set. What we are trying to prove is that the inverse and the identity is in our set, we don’t have to prove that they exist. That’s given to us because we started with a group. So, $a^{-1}$ is out there. It exists. I know what it looks like. I just have to convince myself that it is actually in my subset.

Thus, we can see that Dr. Bond used a group’s conjecture (a subset of a group will be a subgroup if the subset is closed and the inverse of each element of the subset is a member of the subset) to address questions the class had about the identity of a subgroup and the inverses of elements in a subgroup. She did this by directing the students to the fact that the subset in question was a subset of a group, and by reminding the students of the uniqueness of the identity and inverses. However, even after her explanation, there was still evidence that students were unsure as to why the identity of a subgroup needed to be the same as the identity of the group.

Adam: How can you prove that you have inverses, which are like based off of the definition of identity without first saying that you have the identity?

Dr. Bond: Because we have the identity. It exists in $G$ we know what it is. It’s non-negotiable. We can’t change it. We can’t do anything to it.

Adam: But you start with $S$ just being a subset . .

Dr. Bond: A subset of?

Adam: A group.

Dr. Bond: A group, $G$, that has an identity. You can’t just start with an arbitrary set of things and a bigger set of things. That bigger set of things has to be a group, and that’s why when you are talking about subgroups you start with the group. And the minute you say let $G$ be a group and $S$ be a subset of it, that identity is out there and it is unique. Because we have proven that it is unique it is the only option for $S$. . . . The subgroup setting is different because you already have a group structure. You have a unique identity, every element has a unique inverse element, and they are non-negotiable. They are not flexible. They cannot change. They have to remain the same.

We can see that the students in Dr. Bond’s class expressed that they were not convinced that the identity of the subgroup had to be the same as the identity of the group or that the inverses of elements in a subgroup needed to be the same as the inverses of those elements in the group. However, instead of encouraging the students to investigate these statements for themselves, we see that Dr. Bond chose to lecture on this material, making frequent references to the fact that, in a group, inverses and the identity element are unique.

Because Dr. Bond did adjust her lesson in response to the students’ concerns, in that she took more time on this discussion then she had planned, we do consider this to be an example of generative listening. However, it is a somewhat limited example and was ultimately unsuccessful. Again, we argue that Dr. Bond was constrained by her knowledge of content and students. On the one hand, it seems clear that Dr. Bond was aware that the students were struggling to make sense of how proving a subset is a subgroup differs from proving that a set/operation pair forms a group. More specifically, her comments suggest that she was aware that the students needed to understand that in the subgroup case the task is to prove that the identity element is in the subset, whereas in the group case the task is to prove the existence of an element satisfying the identity property.

Indeed, during the debriefing session following this class Dr. Bond reflected on this distinction and the difficulty the students were having with it.

Dr. Bond: I wanted to spend some time on, and this was in the materials, that when you are doing subgroups you don’t have to prove the existence of identities and inverses . . . So I guess I would say that I think it’s important that you know uniqueness of inverses before this point. Because it’s nice to know that this element has a unique inverse out there and we need to know that that guy is in our subset for it to be a subgroup. You know, and it’s not negotiable.

So I hit that pretty hard today, and I think that was important. It’s hard for them. I could tell.

However, Dr. Bond did not seem to be able to completely make sense of the students’ struggles. Specifically, if we restrict our attention to the students’ issues with the identity element, Dr. Bond did not seem to recognize the complexity involved in coming to understand that (1) the identity element of a group will always satisfy the identity property in a subset and (2) only the identity element of the group can satisfy the identity property in a subset.

Although the students repeatedly questioned whether the identity element of the ambient group was the only possibility for the identity element of the subgroup, Dr. Bond did not seem to realize that this was an issue that required an intervention beyond simply referencing the uniqueness of the identity of a group. Ultimately, she did expand her explanation, and analyzing this expanded explanation gives us insight into exactly which aspects of Dr. Bond’s mathematical knowledge for teaching may have been constraining/supporting her ability to make sense of and appreciate the students’ concerns.
Dr. Bond: If you think about your arguments [for why the identity is unique], even if there were two elements that acted like an identity for some element or some subset of group elements, you can still make your proof by contradiction. The identity is unique, so what’s important here is that we know the identity exists in the group and we are not changing the operation. So, that’s our identity, okay. What we need to focus on is, is it in our subset?

This explanation makes it clear that Dr. Bond was aware that things are a bit more complex than her earlier allusions to the uniqueness of the identity may have suggested. The uniqueness of the identity element of a group does not directly imply that the identity of a subgroup must be the same as the identity of the ambient group. However, one can observe that the typical proof (by contradiction) that the identity element is unique does not require an assumption that two elements act as the identity element with respect to all elements of the group in order to reach a contradiction—it is sufficient to assume that two elements act as identity elements with respect to a single element. So, this line of reasoning does obtain a contradiction if an element other than the identity of the ambient group is assumed to act as an identity element with respect to the elements of a subset. Thus, the fact that the identity element of a subgroup must be the identity of the ambient group is a corollary of the proof that the identity is unique and not of the statement that the identity is unique. This is the line of reasoning Dr. Bond was making reference to in her explanation, and is an example of the kind of sophisticated common content knowledge (Ball et al., 2008) we would expect from a mathematician.

However, Dr. Bond’s inability to realize that her argument (that the identity of a subgroup must be the identity of the ambient group) was difficult for students indicates a limitation in her knowledge of content and students. As noted, the students should not have found her initial references to the uniqueness of the identity immediately convincing. Her somewhat elaborated explanation, while perhaps sufficient to convince an expert, would have required some rather quick and sophisticated thinking on the part of her students in order for them to understand it and find it convincing. First, they would have needed to recall the proof that the identity of a group is unique. Then they would have needed to complete the reconstruction of the proof that Dr. Bond mentioned (she did not say explicitly how they would have to modify the proof). They would have needed to do this all of this on the fly since no time was set aside for them to engage in this reasoning. Further, it is unlikely that students at this early point in their study of advanced mathematics would even be aware that a proof can have a corollary (or that it is sometimes valuable to consider whether a proof can be modified to prove a slightly stronger statement). Therefore, in this example we can see that while Dr. Bond had strong common content knowledge, she either did not have, or was not able to draw on, this kind of knowledge of content and students. In this case, it appears that she was unaware of how complex her reasoning would be from the perspective of the students in her class.

5. Conclusions

These three classroom episodes serve to illustrate how Dr. Bond’s mathematical knowledge for teaching impacted her ability to listen to her students. Given Dr. Bond’s strong mathematical background, it may have been presupposed that she would have strong common content knowledge. Not only were we able to identify specific instances that reflected this knowledge, we were able to identify how this type of knowledge was used as Dr. Bond attempted to engage in interpretative and generative listening. For instance, in Episode 2 Dr. Bond drew on her common content knowledge to generalize and test a student’s coset criteria conjecture, and in Episode 3 Dr. Bond drew on her common content knowledge to reason about why the identity of a subgroup must be the same as the identity of the ambient group.

Perhaps more surprising, given Speer and Wagner’s (2009) finding that mathematicians may struggle to make sense of the mathematics posed by their students, was Dr. Bond’s ability to engage with the mathematics of her students in the moment, as she did in Episode 2. Dr. Bond did not immediately recognize the student’s conjecture as equivalent to the statement that each element of a quotient group has order less than or equal to two. However, on the fly she was able to make sense of the conjecture and lead the students in testing it. Along the way, she also figured out that the conjecture was equivalent to a statement about the orders of the elements of the quotient group. We see this as evidence of specialized content knowledge. Thus, while we agree with Speer and Wagner (2009) that mathematicians require additional mathematical knowledge (beyond what they would develop via their professional training), we conjecture that mathematicians may be uniquely well positioned to develop these kinds of knowledge. Perhaps in order to develop specialized content knowledge (for teaching university level mathematics) what mathematicians need most are opportunities to engage with students’ mathematical thinking.

It is pedagogical content knowledge, and knowledge of content and students in particular, that we found to be the primary constraint on Dr. Bond’s ability to listen to her students. In each episode, we can see the students in Dr. Bond’s class clearly expressing difficulties and confusion, and in each of these cases we see Dr. Bond struggling to make sense of their thinking. In Episode 1, Dr. Bond was unable to understand why Adam was viewing left multiplication as being different than right multiplication. In Episode 2, Dr. Bond did not realize that the students were unconvinced by the $Z_4$ counterexample, perhaps because some were not thinking of $Z_4$ as a quotient group. Finally, in Episode 3, Dr. Bond failed to appreciate the complexity

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4 If the identity of the whole group is not in the subgroup, then some other element could be the identity element of the subgroup without violating the uniqueness of the identity in the subgroup, and if the identity of the subgroup is not the identity of the ambient group, then uniqueness of the identity is also not violated for the ambient group.
involved in making the connection between uniqueness of the identity and the fact that the identity of a subgroup must be the identity of the ambient group. Indeed, in each case we found that it was Dr. Bond's lack of knowledge of content and students, or an inability to draw on it in the moment, that restricted her ability to listen interpretively and/or generatively to her students.

6. Discussion

As Speer and Wagner (2009) suggested, by working with mathematicians who have earned doctorates in mathematics, we were able to largely eliminate factors relating to weak content knowledge. As a result, we were able to identify the significance of other domains of knowledge, specifically knowledge of content and students. Our analyses suggest that knowledge of content and students is crucial for supporting teachers in listening productively to students. Additionally, our analyses also allow us to elaborate on Ball et al.'s (2008) description of knowledge of content and students.

In their description of knowledge of content and students, Ball et al. (2008) stated that, central to knowledge of content and students was “knowledge of common student conceptions and misconceptions about particular mathematical content” (p. 401). We see our first episode as an elaboration of this description. In this episode, the key piece of knowledge involves how the students were conceptualizing the operation of combining symmetries, and how this impacted the way they thought about multiplying both sides of an equation by a symmetry. While it is true that students at all levels may think about operations as left to right sequential procedures (Brown et al., 1997; Kieran, 1979; Larsen, 2010), knowledge of this fact alone may be too general to have been helpful to Dr. Bond as she worked to make sense of student thinking. It seems likely that more situated knowledge would have been necessary to support Dr. Bond in listening to her students. In particular she would have benefited from knowing two things, (1) students at this point in the TAAFU curriculum tend to think of the operation of composing symmetries in terms of a left to right sequential procedure, and (2) as a result they would likely view left multiplying by a symmetry as being different than right multiplying by a symmetry. This episode allows us to elaborate Ball et al.'s description in two ways. It points to the importance of not just knowing what the likely conceptions/misconceptions are but also knowing the likely consequences of these conceptions/misconceptions as students engage in new activities. It also suggests that this knowledge needs to be situated in the sense that it is tied to the students' experiences with the curriculum and their previous mathematical activity.

Another component of knowledge of content and students described by Ball et al. (2008) is being able to “anticipate what students are likely to think and what they will find confusing” (p. 401). We see our second episode as suggesting a refinement of this component. In this episode, the key piece of knowledge had to do with the current state of the students' understanding of the quotient group concept and the $\mathbb{Z}/n\mathbb{Z}$ groups. Somehow Dr. Bond needed to realize that neither of these understandings was likely to have been developed enough for the students to be convinced that $\mathbb{Z}/4\mathbb{Z}$ provided a counterexample to their conjecture. Given time, the students probably could have convinced themselves that the $\mathbb{Z}/n\mathbb{Z}$ groups worked much the same way as the partitions of $D_8$ that they had been investigating (and so they were also quotient groups). However, Dr. Bond did not provide the students the time needed to think this through. This episode offers an elaboration of knowledge of content and students in the form of a subcategory focusing on knowledge about how students will perceive examples. In providing a counterexample, the goal is to demonstrate that not all objects in a class have the property in question. For this to be effective, students need to engage in two tasks: recognizing that the counterexample is an element of the class, and seeing that the counterexample fails to have the property. A teacher needs to understand the likely state of the students' current conceptions of the class of objects, the proposed counterexample, and the property under consideration in order to ascertain the likely effectiveness of introducing a counterexample.

Ball et al. (2008) also stated that knowledge of content and students is tied to giving students a task and being able to “anticipate what students are likely to do with it and whether they will find it easy or hard” (Ball et al., 2008, p. 401). We see our final episode as being related to this description. In the third episode the key piece of knowledge has to do with the level of complexity of a given piece of reasoning relative to the mathematical development of the students. Dr. Bond responded to students' questions (about whether the identity of a subgroup had to be the identity of the ambient group) at first with an unelaborated nod to a related result (that the identity of a group is unique) and then later by suggesting that the proof that the identity is unique could serve to prove that the identity of a subgroup must be the identity of the ambient group. Making sense of these arguments requires fairly sophisticated reasoning, and the students would have needed to make sense of this on the fly in order to resolve their concerns. Somehow, Dr. Bond needed to be able to examine her reasoning from the students' perspective to consider whether (and how) they could make sense of it themselves. This episode suggests that it is important for teachers to be able to examine a potential explanation from the perspective of the students in order to determine whether it is likely to be understandable or helpful given their current state of mathematical development.

We conjecture that these categories of knowledge of content and students are relevant beyond the context of undergraduate mathematics. Indeed, each of these categories seem likely to be important in any classroom in which students are actively involved in the development of the mathematical ideas. Further research could consider the wider applicability of these categories.

Our analyses suggest that knowledge of content and students is crucial for supporting teachers in listening productively to students. In each of the three episodes we presented, Dr. Bond's ability to listen interpretively and/or generatively in the moment was constrained by her knowledge of content and students. These categories of listening are particularly significant in classrooms in which the classroom participants are tasked to “lay down a mathematical path as they go, rather than
follow a well-trodden trajectory” (Yackel et al., 2003, p. 117). However, to support teachers in productively listening to their students in the moment, the mathematics education community needs to do more than identify categories of knowledge of content and students. We need to draw on research findings regarding students’ thinking and learning (one could argue these represent the mathematics education field’s knowledge of content and students) and figure out how to get the most useful of this knowledge into the hands of teachers in a way that helps them make sense of their students’ thinking.

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