Calculus II Exam 1 Name:

Directions: Show all your work for full credit. No calculators are needed and thus are not allowed.

1. Derive the derivative of \( y = \sin^{-1} x \) as a function of \( x \).

Solution

\[ y = \sin^{-1} x \text{ means } \sin y = x \]

Now we differentiate implicitly:

\[ \cos y \frac{dy}{dx} = 1 \]

So,

\[ \frac{dy}{dx} = \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1 - x^2}} \]

2. Find the derivative of

\[ f(x) = \ln \frac{x^3 \sqrt{3x^2 - 2}}{x - 1} \]

Solution

\[ f(x) = \ln \frac{x^3 \sqrt{3x^2 - 2}}{x - 1} = 3 \ln x + \frac{1}{2} \ln(3x^2 - 2) - \ln(x - 1) \]

Now we differentiate:

\[ f'(x) = \frac{3}{x} + \frac{6x}{2(3x^2 - 2)} - \frac{1}{x - 1} \]
3. Integrate

\[ \int 3^{\tan x} \sec^2 x \, dx \]

**Solution**

\[ \int 3^{\tan x} \sec^2 x \, dx = \int 3^u \, du \]

where \( u = \tan x \) and so \( du = \sec^2 x \, dx \)

\[ = \frac{1}{\ln 3} 3^u + C = \frac{1}{\ln 3} 3^{\tan x} + C \]

4. Integrate

\[ \int \frac{x^4 - 7x^3 + 13x^2 - 2x - 3}{x^2 - 3x + 1} \, dx \]

**Solution**

\[ \int \frac{x^4 - 7x^3 + 13x^2 - 2x - 3}{x^2 - 3x + 1} \, dx = \int x^2 - 4x + \frac{2x - 3}{x^2 - 3x + 1} \, dx \]

(By long division)

\[ = \frac{x^3}{3} - 2x^2 + \int \frac{du}{u} \]

where \( u = x^2 - 3x + 1 \)

\[ = \frac{x^3}{3} - 2x^2 + \ln |u| + C = \frac{x^3}{3} - 2x^2 + \ln |x^2 - 3x + 1| + C \]
5. Integrate

\[
\int \frac{2x^2}{1 + \sqrt{4x^3}} \, dx
\]

Solution

\[
\int \frac{2x^2}{1 + \sqrt{4x^3}} \, dx = \frac{1}{3} \int \frac{u - 1}{u} \, du
\]

where \( u = 1 + \sqrt{4x^3} \) and so \( du = \frac{1}{2}(4x^3)^{-1/2} \cdot 12x^2 \, dx \)

\[
= \frac{1}{3}(u - \ln |u|) + C = \frac{1}{3}(1 + \sqrt{4x^3} - \ln |1 + \sqrt{4x^3}|) + C
\]

6. Use logarithmic differentiation to find \( f'(x) \), for

\[
f(x) = (2x - 3)^{x^2 - 4}
\]

Solution \( y = (2x - 3)^{x^2 - 4} \) So, we take the log of both sides to get: \( \ln y = \ln(2x - 3)^{x^2 - 4} = (x^2 - 4) \ln(2x - 3) \)

Now, differentiate implicitly:

\[
\frac{1}{y} \frac{dy}{dx} = 2x \ln(2x - 3) + \frac{2(x^2 - 4)}{2x - 3}
\]

SO,

\[
\frac{dy}{dx} = (2x \ln(2x - 3) + \frac{2(x^2 - 4)}{2x - 3})(2x - 3)^{x^2 - 4}
\]
7. Find the equation of the line tangent to the curve \( f(x) = \sin^{-1}(2x^2) \) at the point where \( x = 0 \).

\textbf{Solution} \ From \#1 \ we \ have:

\[
f'(x) = \frac{1}{\sqrt{1 - (2x^2)^2}} \cdot 4x
\]

So \( m = f'(0) = 0 \) and \( x = 0 \) implies that \( y = \sin^{-1}(0) = 0 \) So the equation of the line is:

\[ y = 0. \]

8. A population grows at a rate proportional to its present amount. Suppose that, at time \( t = 0 \), there are 1500 individuals and the rate of change is 15 individuals/day. How long will it take for the population to reach 3500?(Your answer should be in exact not a decimal approximation.)

\textbf{Solution} \ The differential equation is:

\[
\frac{dP}{dt} = kP
\]

Its solution is:

\[ P(t) = Ce^{kt} \]

So \( C = 1500 \) and \( P'(0) = 15 \) gives:

\[ 15 = 1500ke^0 \]

So \( k = \frac{1}{100} \) So \( P(t) = 1500e^{\frac{t}{100}} \) So to find the time until the population doubles is:

\[ 3500 = 1500e^{\frac{t}{100}} \]

\[ \frac{7}{3} = e^{\frac{t}{100}} \]

So

\[ \ln \frac{7}{3} = \frac{1}{100} t \]

So \( t = 100 \ln \frac{7}{3} \)
9. Find the derivative of

\[ f(x) = e^{2x^2+3} + \sin(2x + 5) \]

Solution \( f'(x) = e^{2x^2+3}(4x) + \cos(2x + 5) \cdot 2 \)

10. Integrate:

\[ \int \frac{2t^3}{t^2 + 2} dt \]

Let \( u = t^2 + 2 \) then \( du = 2tdt \) Then

\[ \int \frac{u - 2}{u} dt = u - 2 \ln|u| + C = t^2 + 2 - 2 \ln(t^2 + 2) + C \]