CALCULUS II - EXAM 3 - FALL 2008
November 18, 2008

Show all work for full credit. No books, notes or calculators are permitted. The point value of each problem is given in the left-hand margin. You have 60 minutes.

(4) 1. a) Sketch the portion of the polar spiral \( r = \theta \) with \( 0 \leq \theta \leq \pi \).

(5) b) Find the slope of the tangent to this curve at the point where \( \theta = \pi/2 \).

\[
\begin{align*}
  x &= r \cos \theta = \theta \cos \theta \\
  y &= r \sin \theta = \theta \sin \theta \\

  \frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\theta \cos \theta + 1 \cdot \sin \theta}{\theta (-\sin \theta) + 1 \cdot \cos \theta} \\

  \text{At } \theta = \pi/2, \quad \frac{dy}{dx} &= \frac{\pi/2 \cdot 0 + 1 \cdot 1}{\pi/2 \cdot 1 + 1 \cdot 0} = -2/\pi \\

(4) c) Set up but do not evaluate an integral representing the length of this curve.

\[
  ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \ d\theta = \sqrt{\theta^2 + 1^2} \ d\theta \\

  \text{Length} = \int_0^\pi \sqrt{1 + \theta^2} \ d\theta
2. a) Sketch the polar curve \( r = 3 \sin(2\theta) \) for \( 0 \leq \theta \leq \pi \).

\[
\text{Area} = \int \frac{1}{2} r^2 \, d\theta = \int_0^{\pi/2} \frac{9}{2} \sin^2(2\theta) \, d\theta
\]

2. b) Set up but do not evaluate an integral representing the area enclosed by one loop of the curve in (a).

3. Find the limit of the sequence or explain why it diverges.

a) \( \left\{ \frac{(-1)^n n}{5n + 2} \right\}_{n=1}^{\infty} \)

\[
an = \frac{(-1)^n n}{5n + 2}
\]

For \( n \) even: \( \lim_{n \to \infty} \frac{n}{5n + 2} \to \frac{1}{5} \)

For \( n \) odd: \( \lim_{n \to \infty} \frac{-n}{5n + 2} \to -\frac{1}{5} \)

Sequence diverges (odd and even indexed terms not tending to the same value).

b) \( \left\{ \frac{\ln n}{5n + 2} \right\}_{n=1}^{\infty} \)

\[
\lim_{n \to \infty} \frac{\ln n}{5n + 2} = \lim_{n \to \infty} \frac{\frac{1}{n}}{5} = 0
\]

Sequence converges to 0.
(12) 4. Determine whether the series converges or diverges. If convergent evaluate the sum.

\[ \sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 2} \]

Part 1. \[ \frac{1}{n^2 + 3n + 2} = \frac{1}{n+1(n+2)} = \frac{1}{n+1} - \frac{1}{n+2} \]

Nth partial sum:
\[ S_n = \sum_{n=1}^{n} \frac{1}{n+1} - \frac{1}{n+2} \]
\[ = \left( \frac{1}{2} - \frac{1}{3} + \frac{1}{3} + \ldots + \frac{1}{n+1} \right) - \left( \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n+2} \right) \]
\[ = \frac{1}{2} - \frac{1}{n+2} \rightarrow \frac{1}{2} \text{ as } n \rightarrow \infty \]

a. Series converges:
\[ \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2} = \frac{1}{2} \]

b. \[ \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1}}{5^n+2} = \sum_{n=0}^{\infty} \left( \frac{-3^2}{5} \right)^n \frac{3}{5} \]
\[ = \sum_{n=0}^{\infty} \frac{3}{25} \left( \frac{-9}{5} \right)^n \]

Geometric series: \[ a = \frac{3}{25}, \quad r = -\frac{9}{5} \]
Series diverges since \[ |r| = \frac{9}{5} > 1 \]

(7) 5. Express the decimal \( 0.35 = 0.353535\ldots \) as a geometric series and hence write it as a ratio of integers.

\[ 0.353535\ldots = \frac{35}{100} + \frac{35}{100^2} + \frac{35}{100^2} + \ldots = \sum_{n=1}^{\infty} \frac{35}{100} \left( \frac{1}{100} \right)^{n-1} \]

Geometric series: \[ a = \frac{35}{100}, \quad r = \frac{1}{100} \]

Converges (since \( |r| = \frac{1}{100} < 1 \)) to \[ a \left( \frac{1}{1-r} \right) = \frac{35}{100} \left( \frac{1}{1-\frac{1}{100}} \right) = \frac{35}{99} \]
6. Determine whether the following series converge or diverge. State clearly which test you are using and implement the test as clearly as you can (each answer is worth 2 points, its justification 5 points).

(7) a) \[ \sum_{n=2}^{\infty} \frac{1}{n \sqrt{\ln n}} \] 
Integrated Test

\[ \int_{2}^{\infty} \frac{1}{x \sqrt{\ln x}} \, dx = \lim_{b \to \infty} \int_{2}^{b} \left( \ln x \right)^{-\frac{1}{2}} \frac{1}{x} \, dx = \lim_{b \to \infty} \left[ \frac{2}{\ln x} \right]^{b}_{2} \]

\[ = \lim_{b \to \infty} \frac{2}{\ln b} - \frac{2}{\ln 2} = +\infty \]

Since \( f(x) = \frac{1}{x \sqrt{\ln x}} \) is positive and decreasing for \( x \geq 2 \) and \( \int_{2}^{\infty} \frac{1}{x \sqrt{\ln x}} \, dx \) diverges,

so \[ \sum_{n=2}^{\infty} \frac{1}{n \sqrt{\ln n}} \] diverges by the integral test.

(7) b) \[ \sum_{n=1}^{\infty} \frac{3n^2 + 2n + 1}{(5 + n^2)^2} \] 
Limit Comparison Test

\[ a_n = \frac{3n^2 + 2n + 1}{(5 + n^2)^2} \]
\[ b_n = \frac{3n^2}{(n^2)^2} = \frac{3}{n^2} \]

\[ \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{3n^2 + 2n + 1}{3n^4} = \frac{1}{3} \]

\[ \sum_{n=1}^{\infty} \frac{3n^2}{(5n^2 + 1)^2} \] converges

(\( p \)-series \( p=2 \)) so \( \sum_{n=1}^{\infty} \frac{3n^2}{(5n^2 + 1)^2} \) converges by the limit comparison test.

(7) c) \[ \sum_{n=1}^{\infty} \left( \frac{2n+1}{3n+2} \right)^{2n} \] 
Root Test

\[ a_n = \left( \frac{2n+1}{3n+2} \right)^{2n} \]
\[ \lim_{n \to \infty} |a_n|^{\frac{1}{n}} = \lim_{n \to \infty} \left( \frac{2n+1}{3n+2} \right)^{2} = \left( \frac{2}{3} \right)^{2} = \frac{4}{9} < 1 \]

So \( \sum_{n=1}^{\infty} \left( \frac{2n+1}{3n+2} \right)^{2n} \) converges by the root test.
(7) d) \[ \sum_{n=1}^{\infty} \frac{n^3}{5^n} \]

\[ a_n = \frac{n^3}{5^n} \]

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n+1)^3/5^{n+1}}{n^3/5^n} = \lim_{n \to \infty} \frac{(n+1)^3}{n^3} \cdot \frac{5^n}{5^{n+1}} \]

\[ = \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^3 \cdot \frac{1}{5} = \frac{1}{5} < 1 \]

So \[ \sum a_n \] converges by the ratio test.

7. Determine whether the following series converge or diverge. If the series converges is the convergence conditional or absolute. State clearly which tests you are using and implement the test as clearly as you can (each answer is worth 2 points, its justification 5 points).

(7) a) \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = -1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} - \ldots \]

Alternating series with \( b_n = \frac{1}{\sqrt{n}} \) decreasing with \( b_n \to 0 \) as \( n \to \infty \)

Hence \( \sum \frac{(-1)^n}{\sqrt{n}} \) converges by the alternating series test.

\[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \] diverges (p-series with \( p = \frac{1}{2} < 1 \)) so series \( \sum \frac{(-1)^n}{\sqrt{n}} \) converges conditionally.

(7) b) \[ \sum_{n=1}^{\infty} \frac{(-2)^n}{n!} \]

\[ a_n = \frac{(-2)^n}{n!} \]

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim_{n \to \infty} \frac{2^{n+1} \cdot n!}{2^n \cdot (n+1)!} \]

\[ = \lim_{n \to \infty} \frac{2}{n+1} = 0 < 1 \]

So series \( \sum a_n \) converges absolutely by the ratio test.
(5) 8. A student is approximating the sum

\[ S = \sum_{n=1}^{\infty} \frac{1}{n^3} \]

by the tenth partial sum

\[ S_{10} = \sum_{n=1}^{10} \frac{1}{n^3}. \]

Give upper and lower bounds on the size of the error \( S - S_{10} \).

\[ S - S_{10} < \int_{10}^{\infty} \frac{1}{x^3} \, dx = \left. \frac{-1}{2x^2} \right|_{10}^{\infty} = \frac{1}{200} + \frac{1}{20} = \frac{11}{200} \]

\[ S - S_{10} > \int_{11}^{\infty} \frac{1}{x^3} \, dx = \frac{1}{11^2} = \frac{1}{242}. \]

\[ \frac{11}{242} < S - S_{10} = \sum_{n=11}^{\infty} \frac{1}{n^3} < \frac{1}{200} \]

(5) 9. A student wants to approximate the sum

\[ S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \]

by an \( N \)th partial sum

\[ S_N = \sum_{n=1}^{N} \frac{(-1)^n}{n^3}. \]

a) Is \( S_{10} \) an over or an under estimate?

\[ S_{10} = \frac{1}{1^3} - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \ldots + \frac{1}{10^3} \]

\[ \text{Our estimate:} \quad \left( S_{11} = S_{10} - \frac{1}{11^3} < S < S_{10} \right) \]

b) How many terms \( N \) are required to ensure that \( S_N \) is within \( 10^{-6} \) of \( S \)?

\[ |S - S_N| < 10^{-6} = \frac{1}{(N+1)^3} \]

So enough to take \( \frac{1}{(N+1)^3} \leq 10^{-6} \) \( \Rightarrow (N+1)^3 \geq 10^6 \)

\[ N+1 \geq 10^2 \]

\[ N \geq 99 \]

\[ \text{So } N = 99 \text{ will do.} \]