CALCULUS II - FINAL EXAM
December 17, 2008

Show all work for full credit. No books, notes or calculators are permitted. The
point value of each problem is given in the left-hand margin. You have 1 hour and 50 minutes.

\[
\int \sec x \, dx = \ln |\sec x + \tan x| + C, \quad \int \csc x \, dx = -\ln |\csc x + \cot x| + C
\]

\[
\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \left( \frac{x}{a} \right) + C, \quad \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \left( \frac{x}{a} \right) + C
\]

\[
\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \text{arcsec} \left( \frac{|x|}{a} \right) + C
\]

\[
\int \sqrt{a^2 - u^2} \, du = \frac{1}{2} \left( u\sqrt{a^2 - u^2} + a^2 \arcsin \frac{u}{a} \right) + C,
\]

\[
\int \sqrt{u^2 \pm a^2} \, du = \frac{1}{2} \left( u\sqrt{u^2 \pm a^2} \pm a^2 \ln |u + \sqrt{u^2 \pm a^2}| \right) + C
\]

Maclaurin Series:
\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]

\[
\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}
\]

\[
\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}
\]

\[
\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}
\]

\[
\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n + 1}
\]
1. Evaluate the following integrals.

(7) a) \( \int \frac{\ln x}{x^3} \, dx \)

(9) b) \( \int \frac{dx}{(1 + x^2)^{\frac{3}{2}}} \) (Hint: trig. substitution)

(11) c) \( \int \frac{x^2 + 1}{x^2 + x - 6} \, dx \) (Hint: partial fractions)
(9) d) \( \int \sin^3 x \cos^4 x \, dx \)

(7) e) \( \int (x + 3) \sinh x \, dx \)

(9) f) \( \int e^{2x} \sin x \, dx \)
(8) 2. Set up **but do not evaluate** an integral representing the area of the surface obtained by revolving the curve \( y = x^2 \) for \( 0 \leq x \leq 1 \) about the \( y \)-axis.

3. Evaluate the following limits or indicate that they diverge. Show all work.

(7) a) \( \lim_{x \to 0} \frac{e^{3x} + e^{-x} - 2}{\sin(5x)} \)

(7) b) \( \lim_{x \to 0} (1 + 3x)^{1/x} \)

(7) 4. Find the Taylor series for \( f(x) = 1/x \) about \( a = 2 \). (You may use the geometric series formula.)
5. Solve the initial value problem, \(3x \frac{dy}{dx} = 2(1 + x^2)y^4\), \(y(1) = -\frac{1}{2}\).

6. Find the interval of convergence of the power series \(\sum_{n=1}^{\infty} \frac{(x - 3)^n}{\sqrt{n} 2^n}\). (Make clear the status of any end points.)
7. Determine whether the following series diverge, converge absolutely or converge conditionally. State which test you are using and implement the test as clearly as you can.

(7) a) \[ \sum_{n=0}^{\infty} \frac{n^3}{2^n} \]

(7) b) \[ \sum_{n=0}^{\infty} \frac{n + 3}{\sqrt{n^5 + 3}} \]

(7) c) \[ \sum_{n=3}^{\infty} \frac{(-1)^n}{n(\ln n)^3} \]

(5) d) \[ \sum_{n=1}^{\infty} (-1)^n \frac{n}{n + 2} \]
8. Let $T_2(x)$ be the second degree Taylor polynomial for $f(x) = \sin(2x)$ centered at $a = \pi/4$, and $R_2(x) = f(x) - T_2(x)$ the remainder.

(7) a) Calculate $T_2(x)$.

(3) b) Use Taylor’s inequality to give an upper bound on $|R_2(x)|$ valid for any real number $x$.

9. a) Use the geometric series formula to find a power series expansion for $\frac{1}{1 + x^5}$

(4) b) Use part (a) to evaluate $I = \int_0^{0.1} \frac{dx}{1 + x^5}$ as an infinite series.

(2) c) Explain how many terms you would take in (b) to approximate the value of $I$ within $1/10^8$. 
10. a) Use series given on the cover sheet to find the first three nonzero terms of the Maclaurin series for $f(x) = e^{2x} \ln(1 + 3x)$.

b) Use long division to find the first three nonzero terms of the Maclaurin series for

$$f(x) = \frac{1 + x + 2x^2 + 3x^3 + 4x^4 + \ldots}{1 - x + x^2}$$

11. Use a binomial expansion to find the first three nonzero terms of the Maclaurin series for $f(x) = \sqrt{1 - 3x}$.

12. Evaluate the sums

a) $\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{n!}$

b) $\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}}{5^{n-1}}$
13. a) Sketch the curve with parametric equations \(x = 3 + 2\cos t, \ y = -\sin t\) for \(0 \leq t \leq \pi\) (indicate the direction with an arrow).

14. a) Sketch the polar curve \(r = 2\cos(3\theta), \ 0 \leq \theta \leq \pi\).

14. b) Set up but do not evaluate an integral representing the area of one petal of this curve.

14. c) Find the slope of the tangent to the curve in (a) at \(\theta = \pi/3\).