Calculus 2, Exam 1, Sept 22 - 2009

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RECIATION INSTRUCTOR, HOUR

For best results on this test, draw useful things (such as triangles used in trig substitution, elements \(dV\) of volume) and label them appropriately. You don’t have to do this, but it will make it easier for the grader to follow your thought processes if you do, and it makes it easier to justify and assign partial credit. ALSO: all answers must display some justification on your part, in order to receive any credit at all.

(25) Problem 1. Find exact numerical values for each of the following expressions - if such a value exists. If not, briefly indicate why the expression has no numerical value.

(a) \(\sin^{-1}(2)\) doesn't exist, since there is no number \(x\) such that \(\sin x = 2\).

(b) \(\tan^{-1}(\tan(3\pi/4)) = \left[ \frac{-\pi}{4} \right] = \text{unique } x \text{ in the range } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \text{ such that } \tan x = \tan \left( \frac{3\pi}{4} \right) \)

(c) \(\lim_{x \to \infty} \frac{\sqrt{x^4 - 3x + 2}}{2x^2 - 5x + 1} = \lim_{x \to \infty} \frac{\sqrt{1 - 3/x^3 + 2/x^4}}{2 - 5/x + 1/x^2} = \left[ \frac{1}{2} \right] \)

(d) \(\lim_{x \to 0^+} \ln(x)/x\) - 0/0 form. L'Hôpital doesn't apply.

Answer: \(-\infty\) (or you can say: "No numerical value").

(e) \(\lim_{x \to \infty} (1 + \frac{2}{x})^x = \hat{L}\).
\(\ln \hat{L} = \lim_{x \to \infty} x \ln (1 + \frac{2}{x}) = \lim_{x \to \infty} \frac{\ln (1 + \frac{2}{x})}{\frac{1}{x}} \) - 0/0 form

\(\ln \hat{L} = \lim_{x \to \infty} \frac{1 \cdot (-2/x^2)}{(1 + 2/x)(-1/x^2)} = \lim_{x \to \infty} \frac{2}{1 + 2/x} = 2\)

So \(\ln \hat{L} = 2\) and then \[ \hat{L} = e^2 \].

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(20) **Problem 2.** Find the volume obtained from the region bounded by the graph of \( y = \sin x \), the \( x \)-axis, and the line \( x = \pi/2 \), by revolving the region about the \( y \)-axis.

\[
\begin{align*}
\, dV & = 2\pi rh\,dx = 2\pi x \sin x\,dx \\
V & = 2\pi \int_0^{\pi/2} x \sin x\,dx \\
& = 2\pi \left[ \sin x \right]_0^{\pi/2} + 2\pi \int_0^{\pi/2} \cos x\,dx \\
& = 2\pi \cdot \frac{\pi}{2} + 2\pi \sin x \bigg|_0^{\pi/2} \\
& = \boxed{\pi^2 + 2\,\pi}.
\end{align*}
\]
(10) Problem 4.

(a) Write down the form (without solving for the constants that are involved in this form) of a partial fraction decomposition of the following rational function.

\[ f(x) = \frac{x^4 + 1}{(x^2 + 4)^2(x - 1)^2} \]

\[ \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{Cx + D}{x^2 + 4} + \frac{Ex + F}{(x^2 + 4)^2} \]

(or in some other order).

(b) Evaluate the given integral.

\[ \int \frac{x^4 + 1}{x^2 - 1} \, dx = I \]

\[ \int \frac{x^4 + 1}{x^2 - 1} \, dx = \int \frac{x^4 + 1}{x^2 - 1} \, dx = \int \frac{x^2 + 1 + \frac{2}{x^2 - 1}}{x^2 - 1} \, dx \]

\[ \Rightarrow \quad \frac{x^2 + 1}{x^2 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1} \]

\[ 2 = A(x + 1) + B(x - 1). \]

Take \( x = -1 \), so \( B = -1 \)

Take \( x = 1 \), so \( A = 1 \).

\[ I = \int \left( \frac{x^2 + 1}{x - 1} - \frac{1}{x + 1} \right) \, dx \]

\[ = \frac{1}{3} x^3 + x + \ln |x - 1| - \ln |x + 1| + C \]
(15) **Problem 5.** Determine whether the following "improper" integral $I$ converges, and, if it does, determine the value to which it converges.

\[ I = \int_{1}^{\infty} x e^{-x} \, dx \]

**Parts:** \( u = x \), \( dv = e^{-x} \, dx \)
\[ du = dx \quad v = -e^{-x} \]

\[ I = (uv - \int v \, du)_{1}^{\infty} = -xe^{-x} \bigg|_{1}^{\infty} + \int_{1}^{\infty} e^{-x} \, dx \]

\[ = -xe^{-x} \bigg|_{1}^{\infty} - e^{-x} \bigg|_{1}^{\infty} \]

\[ \lim_{x \to \infty} x e^{-x} = 0, \quad \lim_{x \to \infty} -xe^{-x} = \lim_{x \to \infty} \frac{-x}{e^{x}} = 0 \]

(we use L'Hôpital)

\[ I = (0 - (-e^{-1})) - (0 - e^{-1}) = 2e^{-1} \]
Problem 6. Find the volume obtained by revolving the region bounded by the graph of $y = \sin x$, and the lines $y = 0$ and $x = \pi/2$, about the line $y = 2$.

\[
\begin{align*}
\frac{dV}{dx} &= \pi \left( R^2 - r^2 \right) dx \\
&= \pi \left( 4 - (2 - \sin x)^2 \right) dx \\
&= \pi \left( 4 - (4 - 4\sin x + \sin^2 x) \right) dx \\
&= (4\pi \sin x - \pi \sin^2 x) dx \\
V &= 4\pi \int_0^{\pi/2} \sin x \, dx - \pi \int_0^{\pi/2} \sin^2 x \, dx \\
&= 4\pi (-\cos x) \bigg|_0^{\pi/2} - \pi \left. \frac{1}{2} \left( x - \sin x \cos x \right) \right|_0^{\pi/2} \\
&= \frac{4\pi R^2}{2} - \frac{\pi^2}{4} \\
&= 4\pi C - \frac{\pi^2}{4}
\end{align*}
\]
Calculus 2, Exam 2, October 20 2009

Name

Recitation Instructor and Hour

In order to receive full credit (or any credit it all), answers must be justified, so that they won't be confused with guesses. Drawings (correctly labeled) that bear on a given problem are good sources of partial credit.

(12) Problem 1. Consider the graph of \( y = x^3 / 3 \) from \( x = 0 \) to \( x = 1 \). Set up (and do not attempt to evaluate) the integral which gives the length of the curve.

\[
\begin{align*}
\frac{\text{d}s}{\text{d}x} &= \sqrt{\left(\frac{\text{d}x}{\text{d}x}\right)^2 + \left(\frac{\text{d}y}{\text{d}x}\right)^2} \\
&= \sqrt{1 + \left(\frac{\text{d}y}{\text{d}x}\right)^2} \\
&= \sqrt{1 + x^2} \\
\text{Here } \frac{\text{d}y}{\text{d}x} &= x^2, \text{ so } \\
\frac{\text{d}s}{\text{d}x} &= \sqrt{1 + x^2} \\
S &= \int_{0}^{1} \sqrt{1 + x^2} \, \text{d}x
\end{align*}
\]
Problem 2. On the planet Solaris, the gravitational acceleration at (and near) the surface is \( g \) (in \( \text{kg/sec}^2 \)). The planet is covered by an "ocean" (whose actual substance is the subject of much controversy) with density \( \rho \), measured in \( \text{kg/m}^3 \). The explorers from our own planet (whichever that is) have placed a scientific instrument in the ocean. This device is in the shape of a half-cylinder, sliced lengthwise, and hence with a semi-circular cross section. The radius of the cross section is 4 meters, and the device is suspended with its rectangular side upward and at a depth of 6 meters.

(a) Find the total force due to the pressure of the ocean on the rectangular face, assuming that the face is square. (This is the upper face, at a constant depth of 6 meters).

\[
F = \rho g \cdot \text{depth} \cdot \text{area}
\]

\[
= \rho g \cdot 6 \cdot 8^2
\]

(b) Find the total force due to the pressure of the ocean on one of the semicircular faces. (Hint: You can spare yourself a lot of time by recognizing that part of the integration problem here yields the area of a quarter circle - or of a semicircle - depending on how you set it up.)

\[
dF = \rho g \cdot \text{depth} \cdot dA
\]

\[
= \rho g \left( 6 - y \right) 2 \sqrt{16 - y^2} \, dy
\]

\[
F = 2 \rho g \int_{-4}^{0} \left( 6 - y \right) \sqrt{16 - y^2} \, dy
\]

\[
= 12 \rho g \int_{-4}^{0} \sqrt{16 - y^2} \, dy
\]

\[
= 48 \rho g \pi + \rho g \left( \frac{1}{2} \sqrt{u} \right) \bigg|_{0}^{4} = 48 \rho g \pi + \frac{1}{2} \rho g
\]
(20 = 8 + 12) **Problem 3.** Consider the graph of \( y = \sin x \), from \( x = 0 \) to \( x = \pi/2 \).

(a) Set up an integral which represents the area \( S \) of the surface obtained by revolving the graph about the \( x \)-axis.

\[
dS = 2\pi r \, ds
\]

\[
ds = \sqrt{1 + \cos^2 x} \, dx, \quad r = \sin x
\]

\[
S = 2\pi \int_0^{\pi/2} \sin x \sqrt{1 + \cos^2 x} \, dx
\]

(b) Now go ahead and do the integration, so as to get an exact numerical value for \( S \).

(You will probably need to use one of the "reduction formulas" on the attached "formula page", near the end of this procedure.)

\[
u = \cos x, \quad du = -\sin x \, dx, \quad \sin x \, dx = -du
\]

\[
S = -2\pi \int_0^{\pi/2} \sqrt{1 + u^2} \, du = 2\pi \int_0^{\pi/4} \frac{1}{\sqrt{1 + u^2}} \, du
\]

\[
u = \tan \theta
\]

\[
du = \sec^2 \theta \, d\theta
\]

\[
\frac{1}{\sqrt{1 + u^2}} = \sec \theta
\]

\[
S = 2\pi \int_0^{\pi/4} \sec^3 \theta \, d\theta
\]

\[
= \pi \left( \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right) \bigg|_0^{\pi/4}
\]

\[
= \pi \left( \sqrt{2} + \ln (\sqrt{2} + 1) \right)
\]
(15) Problem 4. Find the centroid of the region \( \mathcal{R} \) bounded below by the \( x \)-axis, and above by the parabola \( y = 4 - x^2 \).

Symmetry \( \Rightarrow \overline{x} = 0 \),

(whence \( \mathcal{C} = (\overline{x}, \overline{y}) \)).

May take \( \rho = 1 \) (density).

- so mass = area,

\[
m = \frac{1}{2} \int_{0}^{2} (4-x^2) \, dx = 2 \left( 4x - \frac{1}{3} x^3 \right) \bigg|_{0}^{2} = 2 \left( 8 - \frac{8}{3} \right) = \frac{32}{3}
\]

\[
dM_x = \frac{1}{2} \left[ (4-x^2)^2 - 0^2 \right] \, dx = \frac{1}{2} \left( 16 - 8x^2 + x^4 \right) \, dx
\]

\[
M_x = \frac{32}{3} \cdot \frac{1}{2} \int_{0}^{2} (16 - 8x^2 + x^4) \, dx
\]

\[
= \left( 16x - \frac{8}{3} x^3 + \frac{1}{5} x^5 \right) \bigg|_{0}^{2}
\]

\[
= 32 - \frac{64}{3} + \frac{32}{5}
\]

\[
\overline{y} = \frac{M_x}{m} = \left( 32 - \frac{64}{3} + \frac{32}{5} \right) / \left( \frac{32}{3} \right)
\]

\[
= 3 - 2 + \frac{3}{5} = \frac{8}{5}
\]

\( \mathcal{C} = (0, \frac{8}{5}) \)
(5) Problem 5. Skip this problem.

(9) Problem 6. Let \( C \) be the parametric curve given by \( x = \sin t \) and \( y = \cos^2 t \), as \( t \) goes from \(-\pi/2\) to \( \pi/2 \). Eliminate the parameter \( t \), and sketch the curve, indicating the direction given by the (increasing) parameter.

\[ x^2 = \sin^2 t \quad \text{so} \quad x^2 + y = 1 \]

(or \( y = 1 - x^2 \))

parabola

\( t = -\pi/2 \) gives
\( x = -1, \; y = 0 \)

\( t = 0 \) gives
\( x = 0, \; y = 1 \)

\( t = \pi/2 \) gives
\( x = 1, \; y = 0 \)
Problem 7. Find the area of the region bounded below by the \(x\)-axis, and above by the semi-ellipse \(x = 2r \cos t, y = r \sin t\), as \(t\) goes from 0 to \(\pi\).

\[
\begin{align*}
\int dA &= y \, dx = y \frac{dx}{dt} \, dt \\
&= r \sin t \left( -2r \sin t \right) \, dt \\
&= -2r^2 \sin^2 t \, dt
\end{align*}
\]

Read from left to right —

\[
A = \int_{\pi}^{0} (-2r^2 \sin^2 t) \, dt = 2r^2 \int_{0}^{\pi} \sin^2 t \, dt
\]

\[
= 2r^2 \frac{1}{2} \int_{0}^{\pi} (1 - \cos 2t) \, dt
\]

\[
= r^2 \left[ t - \frac{1}{2} \sin 2t \right]_{0}^{\pi} = \pi r^2
\]
Problem 8. Let $C$ be the cycloid, given parametrically by $x = \theta - \sin \theta$ and $y = 1 - \cos \theta$. Find an equation for the tangent line to the cycloid at the point where $\theta$ is equal to $\pi/4$.

\[
\text{Slope} \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{1 - \cos \theta}{\sin \theta}
\]

At $\theta = \pi/4$:

\[
\frac{dy}{dx} = \frac{1 - \sqrt{2}/2}{\sqrt{2}/2} = \sqrt{2} - 1
\]

Also, at $\theta = \pi/4$ get $x = \pi/4 - \sqrt{2}$

and $y = 1 - \sqrt{2}/2$.

Point-slope equation for tangent line:

\[
y - (1 - \sqrt{2}/2) = (\sqrt{2} - 1)(x - (\pi/4 - \sqrt{2}/2))
\]
Calculus 2, Exam 3, November 17 2009

Identify yourself, your recitation instructor, and your recitation hour, at the top of the page. In order to receive full credit (or any credit it all), answers must be justified.

(7=2+2+3) Problem 1. Consider the sequence $a_1 = 1.01$, $a_2 = 1.01001$, $a_3 = 1.010010001$, and so on (so that at the $n^{th}$ step one attaches an extra string of $n$ zeros followed by 1).

(a) Is the sequence increasing?  
For sure.

(b) Does the sequence have an upper bound? If so, name one such upper bound.

(c) Does the sequence converge? Briefly explain. Basic theorem: any sequence which is monotone and bounded must converge.

(20 = 4x5) Problem 2. Your task is to decide if the given sequence $\{a_n\}_{n=1}^\infty$ converges, or whether it diverges. If it converges, determine the number to which it converges.

(a) $a_n = \frac{8^n}{9^n}$.

\[ \lim_{n \to \infty} a_n = \left\lfloor 0 \right\rfloor \quad \left( \lim_{n \to \infty} r^n = 0 \text{ if } r < 1 \right). \]

(b) $a_n = n \sin(1/n)$. Pretend $n$ is a continuous variable.

Apply L'Hôpital to $f(n) = \frac{\sin(1/n)}{1/n}$ (O/D-form)

Get $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \cos(1/n) = \cos 0 = 1$.

(c) $a_n = \sqrt{2n^3 - 5n^2 + n - 7} / (n + 4)$.

Diverges (degree in numerator = 3/2 is greater than that of denominator)

(d) $a_n = \cos(n) / \ln(n)$ (and where $n$ goes from 2 to $\infty$).

\[ \lim_{n \to \infty} a_n = 0 \]

since $|\cos n| \leq 1$ while $\ln(n) \to \infty$. 

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(18) Problem 3. Your next task is to decide, for each of the following series, whether or not the series converges and, if it does converge, to determine the number that it converges to.

(a) \[
\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{5^n} = \sum_{n=0}^{\infty} \left( -\frac{2}{5} \right)^n = \frac{1}{\left( 1 + \frac{2}{5} \right)}
\]

(b) \[
\sum_{n=2}^{\infty} \frac{3^n}{5^n} = \frac{1}{1 - \frac{3}{5}} - \left( 0^{th} + 1^{st} \text{ terms of geometric series} \right)
\]
\[
= \frac{1}{\frac{5}{2} - 1 - \frac{3}{5}}
\]

(c) \[
\sum_{n=1}^{\infty} \frac{2^{n+2}}{3^n} = 8 \sum_{n=1}^{\infty} \left( \frac{2}{3} \right)^n = 8 \left[ \frac{1}{1 - \frac{2}{3}} - 1 \right]
\]
\[
= 8 \left( 3 - 1 \right) = 8
\]

(10) Problem 4. Use the integral test to decide whether the 3/2-series \[ \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \] converges.

\[
\int_{0}^{\infty} x^{-3/2} \, dx = -2x^{-1/2} \bigg|_{0}^{\infty}
\]
\[
= \text{The integral converges since } x^{-1/2} \to 0 \text{ (a number)} \text{ as } x \to \infty.
\]

The integral test then yields convergence of the given series.
(30) Problem 5. Use an appropriate test to determine convergence or divergence for each of the following series. (You should not need to employ the integral test for any of these. You are free to use all of the basic results about geometric series and p-series.)

(a) \( \sum_{n=1}^{\infty} \frac{1}{n^2+1} \)  
\[ \text{Compare with } \sum \frac{1}{n^2} \text{ (convergent, 2-series)} \]
\[ \frac{1}{n^2+1} < \frac{1}{n^2} \]  
so the given series \boxed{\text{converges}}.

(b) \( \sum_{n=1}^{\infty} \frac{n}{(n^2+1)} \)
\[ \lim_{n \to \infty} \frac{n}{n^2+1} \]
\[ = \lim_{n \to \infty} \frac{n^2}{n^2+1} = 1 \text{ (positive number)} \]
so the given series \boxed{\text{converges also}}.

(c) \( \sum_{n=1}^{\infty} \frac{3^n}{(4^n-1)} \)
\[ \lim_{n \to \infty} \frac{3^n}{4^n-1} = \lim_{n \to \infty} \frac{4^n}{4^n-1} = 1 \]
\[ \boxed{\text{Converges}} \]

(d) \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln(n)}{n} \)
\[ \text{Alt. Series Test} \]
\( (1) \ \frac{\ln(n)}{n} \to 0 \) as \( n \to \infty \) (well known, via L'Hôpital's Rule)
\( (2) \ D_x \left( \frac{\ln x}{x} \right) = \frac{1-x \ln x}{x^2} < 0 \) for all large \( x \), so the terms are decreasing.

(e) \( \sum_{n=1000}^{\infty} \frac{1}{n} \)
\[ \lim_{n \to \infty} \frac{1}{n} = 0 = 1 \]
so the series \boxed{\text{converges}.

(by the (aptly named) divergence test).}
(5) **Problem 6.** Find two different expressions in polar coordinates for the point in the plane whose rectangular coordinates are \((1, -1)\).

\[
\begin{align*}
\text{(Many solutions)} \\
(r, \theta) &= (\sqrt{2}, -\pi/4) \\
\text{or} \quad (\sqrt{2}, 3\pi/4) \quad \text{or} \ldots
\end{align*}
\]

(10) **Problem 7.** Roughly sketch the portion of the spiral \(r = \sqrt{\theta}\) (in polar coordinates \((r, \theta)\)), as \(\theta\) goes from 0 to \(\pi\). Find the area of the region bounded above by this curve and below by the line \(\theta = 0\).

\[
\begin{align*}
dA &= \frac{1}{2} r^2 d\theta \\
A &= \frac{1}{2} \int_0^{\pi} \theta \, d\theta \\
&= \frac{1}{4} \theta^2 \bigg|_0^\pi \\
&= \frac{\pi^2}{4}
\end{align*}
\]