Calculus 2, Exam 3, November 16 2010

Identify yourself, your recitation instructor, and your recitation hour

Nessuno \ Nuncio

In order to receive full credit (or any credit it all), answers must be justified.

(2+2+4) Problem 1. Consider the sequence \( \{a_n\}_{n=1}^{\infty} \), where \( a_n = (0.99999999)^n \).

(a) Is the sequence increasing? Decreasing? Neither?

\[ \text{Decreasing} \]

(b) Does the sequence have a lower bound? If so, name one.

\[ \text{Yes} \quad (0 \text{ is a lower bound. So is } -10^6) \]

(c) Does the sequence converge? Can you explain why (or why not)?

Monotone convergence theorem ("brick wall theorem") \( \Rightarrow \) decreasing, bounded sequence converge.

(12) Problem 2. Your task is to decide if the given sequence \( \{a_n\}_{n=0}^{\infty} \) converges, or whether it diverges. If it converges, determine the number to which it converges.

(a) \( a_n = \frac{9^n}{8^n} \)

\[ \text{diverges} \quad (\lim n^n \text{ diverges if } r > 1.) \]

(b) \( a_n = \cos^2(1/\sqrt{n+1}) \)

\( \lim_{n \to \infty} \cos^2 \frac{1}{\sqrt{n+1}} = \cos^2 \left( \lim_{n \to \infty} \frac{1}{\sqrt{n+1}} \right) 
\]

\( = \cos^2 0 = 1 \)

(c) \( a_n = \frac{\sqrt{2n^2-n^2+1}}{n^2+n+1} \)

Ignore lower degree terms: \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{\sqrt{2n^2}}{n^2} = \sqrt{2} \)

(d) \( a_n = 1 - \frac{1}{2} + \cdots + (-1)^n \frac{1}{2^n} \).

\( n \text{th partial sum of geometric series} \sum_{n=0}^{\infty} (-\frac{1}{2})^n \) Converges to \( \frac{1}{1-(-\frac{1}{2})} = \frac{2}{3} \)
(12) Problem 3. This problem concerns the 4-series \( \sum_{n=1}^{\infty} \frac{1}{n^4} \).

(a) Apply the integral test, to show that the series converges.

\[
 f(x) = \frac{1}{x^4} \text{ is positive and decreasing on } [1, \infty), \\
\int_1^\infty x^{-4} \, dx = -\frac{1}{3} x^{-3} \bigg|_1^\infty = \frac{1}{3} x^3 \bigg|_1^\infty = \frac{1}{3}
\]

The integral converges, so the series converges.

(b) From the result of part (a), you know for sure that the given series converges to some number \( s \). Does \( 1 + 1/16 + 1/81 + 1/256 \) give an approximation of \( s \) to within 1/1000? (Justify your answer.)

4th partial sum of the series leaves the "error" term \( R_4 \), where

\[
\sum_{n=3}^{\infty} x^{-4} \, dx < R_4 \leq \int_3^{\infty} x^{-4} \, dx
\]

\[
= \left. \frac{1}{3x^3} \right|_{3}^{\infty} \leq R_4 \leq \frac{1}{3x^3} \bigg|_{\infty}^{\infty} \quad \text{(as in part (a))}
\]

\[
\frac{1}{81} \leq R_4 \leq \frac{1}{3 	imes 64}
\]

The error exceeds \( \frac{1}{81} \), so the answer is NO. The partial sum does not give accuracy to within \( \frac{1}{1000} \).
(30) Problem 4. Use an appropriate test, and clearly indicate which test you are using, in order to determine convergence or divergence for each of the following series. (You should not need to employ the integral test for any of these, and you are free to use all of the basic results about geometric series and p-series when employing a comparison test.)

(a) \( \sum_{n=1}^{\infty} \frac{1}{n^2 + n} \)

\( \frac{1}{n^2 + n} < \frac{1}{n^2} \). So \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges.

(b) \( \sum_{n=2}^{\infty} \frac{2n}{n^2 + n} = \sum_{n=2}^{\infty} \frac{2}{n+1} \)

\( \lim_{n \to \infty} \frac{2}{n+1} \to 2 \) (a positive real number)

so the series diverges.

(c) \( \sum_{n=1}^{\infty} \frac{3^n}{4^n - 1} \)

\( \frac{3^n}{4^n - 1} = \frac{4^n}{4^n - 1} = \frac{1}{1 - \frac{1}{4^n}} \)

\( \lim_{n \to \infty} \frac{1}{1 - \frac{1}{4^n}} \to 1 \)

so the series converges.

(d) \( \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n} - 1} \)

Alt. series test

\( \lim_{n \to \infty} \frac{1}{\sqrt{n} - 1} = 0 \), \( \frac{1}{\sqrt{n} - 1} > \frac{1}{\sqrt{n+1} - 1} \)

converges.

(e) \( \sum_{n=1}^{\infty} (-2)^n \)

\( \lim_{n \to \infty} (-2)^n \neq 0 \), so the series diverges.

(3) (or) can use the root test.)
Problem 5. This problem concerns the power series \( \sum_{n=1}^{\infty} (-1)^n \frac{(x+2)^n}{3^n \sqrt{n}} \).

(a) Find the radius of convergence for this power series.

\[
\frac{|x+2| \frac{n+1}{3^n \sqrt{n}}}{(x+2) \frac{3^{n+1}}{3^{n+1} \sqrt{n+1}}} = \frac{|x+2|}{3} \sqrt{\frac{n}{n+1}}
\]

\[
\lim_{n \to \infty} \frac{|x+2|}{3} = \frac{|x+2|}{3} < 1
\]

Convergence if
\[
|x+2| < 3
\]

\[
\sum (\text{and divergence for } |x+2| > 3)
\]

\[
R = 2
\]

(b) Now "fine-tune" your result, by finding the interval of convergence for the given power series.

When \( x = 1 \), the series becomes \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \) (which converges).

When \( x = -5 \), the series becomes \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) (which diverges).

Interval of convergence is \( (-5, 1] \).
(16) Problem 6. Find the radius of convergence for each of the following power series.
(You aren’t being asked for the interval of convergence.)

(a) \( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} \) (Ratio Test)

\[
\frac{|x|^{2(n+1)-1}}{|x|^{2n-1}} \cdot \frac{(2n-1)!}{(2(n+1)-1)!} = \frac{|x|^{2n+1}}{|x|^{2n-1}} \cdot \frac{(2n-1)!}{(2n+1)!}
\]

\[
= \frac{|x|^{2}}{(2n+1)2n} \xrightarrow[n \to \infty]{} 0 \quad \text{(no matter what number x is)}
\]

Converges for all x. So, \( R = \infty \).

(b) \( \sum_{n=0}^{\infty} n!(x-3)^n \) (Ratio Test)

\[
\frac{|x-3|^{n+1}}{|x-3|^n} \cdot \frac{(n+1)!}{n!} = \frac{|x-3|}{n} \xrightarrow[n \to \infty]{} \infty
\]

for all x except \( x = 3 \).

The series only converges when \( x = 3 \).

\( R = 0 \).

(6) Problem 7. Find the exact value of \( \sum_{n=2}^{\infty} 1/3^n \). (Notice that the summation starts with \( n = 2 \).)

\[
\sum_{n=2}^{\infty} \frac{1}{3^n} = \sum_{n=0}^{\infty} \frac{1}{3^n} - 1 - \frac{1}{3} = \left( \frac{1}{3^0} + \frac{1}{3^1} \right) \text{ (the sum)}
\]

\[
= \frac{1}{1 - \frac{1}{3}} - 1 - \frac{1}{3} = \frac{3}{2} - 1 - \frac{1}{3} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}
\]