Introduction to ODE systems and linear algebra notation

Nathan Albin
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1 The mass-spring equation as a system

Starting from the mass-spring equation

\[ mx'' + cx' + kx = 0, \]

we can generate an equivalent first-order system using the trick we’ve used a few times in lab and lecture. We introduce a new variable, \( y = x' \) and then use the differential equation to discover that

\[
y' = x'' = -\frac{k}{m}x - \frac{c}{m}x' = -\frac{k}{m}x - \frac{c}{m}y.
\]

Putting together the two pieces, we get the system:

\[
\begin{align*}
x' &= y \\
y' &= -\frac{k}{m}x - \frac{c}{m}y
\end{align*}
\]

or, equivalently,

\[
\begin{align*}
x' &= 0x + 1y \\
y' &= -\frac{k}{m}x - \frac{c}{m}y.
\end{align*}
\]

We can also move from the system back to the ODE like this.

\[
x'' = y' = -\frac{k}{m}x - \frac{c}{m}y = -\frac{k}{m}x - \frac{c}{m}x'.
\]

Multiplying through by \( m \) and rearranging gets us back to the second-order equation.

2 Linear algebra notation

You can think of matrices and vectors as providing a nice shorthand notation for certain operations. For example, when we have several variables, \( x \) and \( y \) in our case, we can create a new object (a vector) called \( \vec{x} \) that keeps track of all of them using one symbol:

\[
\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}.
\]

There is also a convenient notation for derivatives using vectors. Namely, we’ll just write

\[
\vec{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}.
\]
It’s important to keep in mind that this is just notation. There’s nothing magical happening here. Now, let’s try to write down the differential equation system using this new notation. An obvious thing to try would be to write

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix}
0x + 1y \\
-\frac{k}{m}x - \frac{c}{m}y
\end{bmatrix}.
\]

(1)

This makes sense as long as we agree about what the = sign means for vectors.

<table>
<thead>
<tr>
<th>Equality of vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>When we write</td>
</tr>
</tbody>
</table>
| \[ \begin{bmatrix}
a \\
b
\end{bmatrix} = \begin{bmatrix}
c \\
d
\end{bmatrix}, \] |
| we mean that \( a = c \) and \( b = d \). |

With that understanding, equation (1) now makes sense and means the same thing as the original system of ODEs. We can further simplify notation by introducing a matrix. Let’s write

\[
A = \begin{bmatrix}
0 & 1 \\
-\frac{k}{m} & -\frac{c}{m}
\end{bmatrix}.
\]

Again, keep in mind that this is just a notation. Now, let’s all agree to the following rule.

<table>
<thead>
<tr>
<th>Matrix-vector multiplication</th>
</tr>
</thead>
<tbody>
<tr>
<td>When we write</td>
</tr>
</tbody>
</table>
| \[ \begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix}, \] |
| we mean the vector \[ \begin{bmatrix}
ax + by \\
cx + dy
\end{bmatrix}. \] |

Taking \( A \) and \( \vec{x} \) from above, notice that, with this new notational agreement,

\[
A\vec{x} = \begin{bmatrix}
0 & 1 \\
-\frac{k}{m} & -\frac{c}{m}
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
0x + 1y \\
-\frac{k}{m}x - \frac{c}{m}y
\end{bmatrix}.
\]

But that last vector is exactly the vector on the right of equation (1)! This provides a beautiful shorthand notation for the mass-spring system.

\[
\vec{x}' = A\vec{x}, \quad \text{where} \quad A = \begin{bmatrix}
0 & 1 \\
-\frac{k}{m} & -\frac{c}{m}
\end{bmatrix}, \quad \vec{x} = \begin{bmatrix}
x \\
y
\end{bmatrix}.
\]

(2)
3 What linear algebra notation can do for you

At the risk of being overly repetitive, let’s remember one more time that there is no new
math here. All we have done is taken concepts we were already working with and expressed
them with a different notation. That means that there is absolutely no conceptual difference
between equations (1) and (2). At this point, you may reasonably wonder “Why bother?”

Well, since linear algebra notation is used almost universally within science and engineer-
ing mathematics, the answer must be that the notation helps us understand and manipulate
the quantities we’re interested in. In fact, you’ve already seen this phenomenon in Calculus.
Consider, for example, the difference between the notations

\[ f''(x) \quad \text{and} \quad \lim_{h \to 0} \frac{f(x+h+k) - f(x+k) - f(x)}{h} \]

These two expressions actually refer to exactly the same quantity (the second derivative of
\( f \) at the point \( x \)). However, the first has a nice form that we have all developed a decent
intuition about, while the second is the stuff of nightmares. The first notation is useful
because it hides the limits from us and allows us to focus on learning helpful rules for
computation, like

\[ (e^{2x} - 3x^2)'' = (2e^{2x} - 6x)' = 4e^{2x} - 6. \]

Similarly, linear algebra notation provides a bunch of easy-to-use computation rules.
You’ll find more of these in the text, and you’ll certainly learn a lot about them in a Linear
Algebra or Matrix Theory and in later classes. Here are a few things that will be helpful for
us.

### Scaling matrices and vectors

Let

\[ \overrightarrow{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

be a vector and a matrix respectively, and let \( s \) be any number (often called a scalar
in linear algebra). Then \( \overrightarrow{x} \) and \( A \) can be scaled by \( s \) using the following definitions.

\[ s\overrightarrow{x} = \begin{bmatrix} sx \\ sy \end{bmatrix} \quad \text{and} \quad sA = \begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}. \]
Adding and negating vectors

Let \( \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \) and \( \vec{y} = \begin{bmatrix} z \\ w \end{bmatrix} \) be two vectors. Then we can form some new vectors using the following definitions.

\[
\vec{x} + \vec{y} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} x+z \\ y+w \end{bmatrix} \quad \text{and} \quad -\vec{x} = (-1)\vec{x} = \begin{bmatrix} -x \\ -y \end{bmatrix}.
\]

(You can also add and negate matrices in a similar way, entry-by-entry.) These rules allow us to write things like the following.

\[
3 \begin{bmatrix} -5 \\ 2 \end{bmatrix} - 8 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} -5 \\ 2 \end{bmatrix} + (-8) \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \times (-5) \\ 3 \times 2 \end{bmatrix} + \begin{bmatrix} (-8) \times 1 \\ (-8) \times 3 \end{bmatrix} = \begin{bmatrix} -15 \\ 6 \end{bmatrix} + \begin{bmatrix} -8 \\ -24 \end{bmatrix} = \begin{bmatrix} -23 \\ -18 \end{bmatrix}.
\]

Matrices, vectors and scalars also satisfy a bunch of algebraic properties, similar to things you learned when working with numbers. (Remember commutativity, associativity, the distributive law, etc.?) Here is a small sample of the useful properties this notation provides. We won’t prove them in this class, but they all basically come down to using the definitions above together with the usual algebraic properties of real numbers.

A few linear algebra properties

In the following, \( \vec{x} \) and \( \vec{y} \) are vectors, \( A \) and \( B \) are matrices, and \( s \) and \( t \) are numbers (scalars).

- \( \vec{x} + \vec{y} = \vec{y} + \vec{x} \)
- \( A + B = B + A \)
- \( (st)\vec{x} = s(t\vec{x}) = t(s\vec{x}) \)
- \( s\vec{x} + s\vec{y} = s(\vec{x} + \vec{y}) \)
- \( s\vec{x} + t\vec{x} = (s+t)\vec{x} \)
- \( s(A\vec{x}) = (sA)\vec{x} = A(s\vec{x}) \)
- \( A(s\vec{x} + t\vec{y}) = sA\vec{x} + tA\vec{y} \)
- \( (sA + tB)\vec{x} = sA\vec{x} + tB\vec{x} \)
4 Connecting linear algebra to ODEs

Now, let’s start from the linear algebra form of a linear system of first-order ODEs.

$$\vec{x}' = A\vec{x} \quad \text{where} \quad A = \begin{bmatrix} 0 & 2 \\ 8 & 0 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

If we write this back in calculus-looking notation, we get the system

$$\begin{cases} x' = 0x + 2y \\ y' = 8x + 0y \end{cases} \quad \text{or} \quad \begin{cases} x' = 2y \\ y' = 8x \end{cases}.$$

This system can be turned quite easily into a second-order equation as follows.

$$x'' = 2y' = 2(8x) = 16x \quad \text{so} \quad x'' - 16x = 0.$$

The characteristic polynomial of this second-order, linear, constant coefficient equation has roots $\pm 4$, so the general solution is

$$x(t) = c_1e^{4t} + c_2e^{-4t}.$$

Then we can use the first equation in the system to find $y(t)$ as

$$y(t) = \frac{1}{2}x'(t) = \frac{1}{2}(4c_1e^{4t} - 4c_2e^{-4t}) = 2c_1e^{4t} - 2c_2e^{-4t}.$$

Now, let’s translate back into linear algebra notation.

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1e^{4t} + c_2e^{-4t} \\ 2c_1e^{4t} - 2c_2e^{-4t} \end{bmatrix}.$$

If we use the rules of vectors and scalars we just learned, we can rewrite this as

$$\vec{x} = \begin{bmatrix} c_1e^{4t} \\ 2c_1e^{4t} \end{bmatrix} + \begin{bmatrix} c_2e^{-4t} \\ -2c_2e^{-4t} \end{bmatrix} = c_1e^{4t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2e^{-4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

So, it looks like the general solution to the system arises by just multiplying the two linearly independent functions $c_1e^{4t}$ and $c_2e^{-4t}$ by two special vectors, and then adding.

What’s so special about these vectors? Many things actually, but here’s a start. Let’s write

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Then

$$A\vec{v}_1 = \begin{bmatrix} 0 & 2 \\ 8 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \times 1 + 2 \times 2 \\ 8 \times 1 + 0 \times 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 4\vec{v}_1,$$

and

$$A\vec{v}_2 = \begin{bmatrix} 0 & 2 \\ 8 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \times 1 + 2 \times (-2) \\ 8 \times 1 + 0 \times (-2) \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \end{bmatrix} = -4\begin{bmatrix} 1 \\ -2 \end{bmatrix} = -4\vec{v}_2.$$

So the vector $\vec{v}_1$ that goes along with the function $e^{4t}$ satisfies $A\vec{v}_1 = 4\vec{v}_1$ and the vector $\vec{v}_2$ that goes along with the function $e^{-4t}$ satisfies $A\vec{v}_2 = -4\vec{v}_2$! Coincidence? Nope. You’ll get to experiment with the implications of these special vectors in the lab and we’ll talk about them more in upcoming lectures.