IMPROVED BOUNDS FOR COMPOSITES AND RIGIDITY
OF GRADIENT FIELDS

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Abstract. We determine an improved lower bound for the conductivity of three-component composite materials. Our bound is strictly larger than the well-known Hashin-Shtrikman bound outside the regime where the latter is known to be optimal. The main ingredient of our result is a new quantitative rigidity estimate for gradient fields in two dimensions.

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1. Introduction

The theory of homogenization permits to determine the macroscopic properties of microstructured materials. This corresponds to determining an appropriate weak limit of PDEs with rapidly oscillating coefficients. In the case of periodic homogenization of scalar problems, one considers periodic microstructures which can be described by a position-dependent conductivity \( \sigma \in L^\infty(Y) \), with \( Y = (0,1)^2 \) the unit square. Throughout the paper, \( \sigma \) will be scalar so that we will be concerned with locally isotropic media. We assume the ellipticity condition \( \sigma \geq \sigma_1 > 0 \) a.e. The homogenized (or macroscopic) conductivity \( \sigma_{\text{hom}} = \sigma_{\text{hom}}^T \in \mathbb{R}^{2\times2} \) can be determined by

\[
\xi \cdot \sigma_{\text{hom}} \xi = \min \left\{ \int_Y \sigma(x) |\nabla u|^2(x) dx : u \in W^{1,2}_{\#,\xi}(Y) \right\}, \tag{1.1}
\]

for every \( \xi \in \mathbb{R}^2 \). Here \( W^{1,2}_{\#,\xi}(Y) \) is the set of \( u \in W^{1,2}(Y) \) such that \( x \mapsto u(x) - \xi x \) is \( Y \)-periodic (in the sense of traces) and with zero mean over \( Y \). It is easy to see that \( \sigma_{\text{hom}} \) is positive definite.

The theory of composites is aimed at understanding which are the possible values of \( \sigma_{\text{hom}} \), with \( \sigma \) varying in some subset of \( L^\infty(Y) \). This rich theory has been the subject of many monographic texts including Bensoussan \textit{et al.} (1978), Sanchez-Palencia & Zaoui (1987), Dal Maso (1993), Jikov \textit{et al.} (1994), Braides & Defranceschi (1998), Cioranescu & Donato (1999), Cherkaev (2000), Milton (2002). In this paper, we focus on the case where the macroscopic conductivity \( \sigma_{\text{hom}} \) is proportional to the identity matrix \( \text{Id} \),

\[
\sigma_{\text{hom}} = s_{\text{hom}} \text{Id}, \quad s_{\text{hom}} > 0. \tag{1.2}
\]
The methods of Hashin and Shtrikman (1962) prove that for every isotropic microstructure $\sigma$ (i.e., for every microstructure obeying (1.2)) one has

$$s_{\text{hom}} \geq s_{\text{HS}} = \left( \int_Y (\sigma + \sigma_1)^{-1} \, dx \right)^{-1} - \sigma_1. \quad (1.3)$$

By duality, there is also a corresponding upper bound on $s_{\text{hom}}$. Since the techniques for treating the dual problem are identical, we treat only the lower bound in the present paper. It is natural to ask whether this bound can be improved, given some class of possible values of $\sigma$. The answer depends of course on the chosen class.

In problems coming from shape optimization one typically deals with mixing a finite number of components, with conductivities $\sigma_1 < \sigma_2 < \cdots < \sigma_N$, and prescribed volume fractions $m_1, m_2, \ldots, m_N \in [0,1]$, with $\sum_{i=1}^{N} m_i = 1$ (Bendsoe & Kikuchi 1988, Kohn & Strang 1986, Cherkaev 2000, Allaire 2002, Milton 2002). The microscopic conductivity $\sigma$ can vary in the set

$$\Sigma = \{ \sigma \in L^\infty(Y) : |\{x \in Y : \sigma(x) = \sigma_i\}| = m_i, i = 1, \ldots, N \} . \quad (1.4)$$

In the case of two materials the bound (1.3) is optimal, in the sense that for any admissible $\sigma_1, \sigma_2, m_1, m_2$ one has

$$s_{\text{HS}} = \inf \{ s_{\text{hom}} : s_{\text{hom}} = s_{\text{hom}} \text{Id for some } \sigma \in \Sigma \} .$$

In fact, in this case the infimum is attained (Tartar 1985; Vigdergauz 1994; Astala & Nesi 2003).

For the rest of this paper, we will focus on the case of three materials ($N = 3$). This case turns out to be substantially more complex, and one has to distinguish several parameter ranges. In the case $m_1 \geq \alpha(1 - m_2)$ with

$$\alpha = \frac{2\sigma_1(\sigma_3 - \sigma_2)}{(\sigma_2 + \sigma_1)(\sigma_3 - \sigma_1)} \quad (1.5)$$

optimality of (1.3) was proven by Milton (1981) by a construction based on “coated spheres”. Later, Lurie and Cherkaev (1985) found different microstructures saturating the bound exactly in the same regime. A corresponding result for anisotropic composites was also given by Milton and Kohn (1988).

In the broader range

$$m_1 \geq \alpha(\sqrt{m_2} - m_2) \quad (1.6)$$

optimality of (1.3) was proven by Gibiansky and Sigmund (2000) with constructions that were inspired by numerical simulations. A simpler proof, using finite-order laminates, which additionally permits treatment of the case of anisotropic composites, was then obtained in Albin (2006) (see also Albin et al., 2007). The question of optimality of (1.3) in the case that (1.6) does not hold remained open.
The classical lower bound (1.3) was improved by Nesi (1995), who proved that
\[ s_{\text{hom}} \geq s_{\text{HS}} + \Delta_N = \max_{\sigma \in [\sigma_1, \sigma_2]} \left\{ \left( \frac{m_1}{2\sigma_1} + \frac{m_2}{\sigma_2 + \lambda} + \frac{m_3}{\sigma_3 + \lambda} \right)^{-1} - \lambda \right\} \] (1.7)
The function \( \Delta_N \) is however strictly positive only in a subset of the range of parameters for which (1.6) does not hold.

Albin, Cherkaev and Nesi (2007) showed that, if (1.6) is not satisfied, then
\[ s_{\text{hom}} > s_{\text{HS}} \] (1.8)
for all \( \sigma \in \Sigma \) with \( C^1 \) smooth level sets (again in the case \( s_{\text{hom}} = s_{\text{hom}} \text{Id} \)).

The question of the existence and regularity of a \( \sigma \in \Sigma \) which minimizes \( s_{\text{hom}} \) in general remains open.

In this paper, we make no smoothness assumption and prove that the Hashin-Shtrikman bound (1.3) can be improved whenever (1.6) does not hold, thereby answering completely the question of optimality of (1.3) for the case of three components.

**Theorem 1.1.** There is a function \( \Delta \) such that for all \( \sigma_i \) and \( m_i \), and all \( \sigma \in \Sigma \) such that \( s_{\text{hom}} = s_{\text{hom}} \text{Id} \), it holds that
\[ s_{\text{hom}} \geq s_{\text{HS}} + \Delta \] (1.9)
with \( \Delta > 0 \) whenever (1.6) is not satisfied.

One key idea, which permits in particular a simple proof of (1.3), is to “vectorialize”. This idea appeared in print for the first time in Tartar (1979) and later in a more detailed version in Tartar (1985) and Lurie & Cherkaev (1984). By considering (1.1) for two vectors \( \xi \) and \( \xi' \in \mathbb{R}^2 \) one obtains
\[ E^T \cdot \sigma_{\text{hom}} E^T = \min \left\{ \int_Y \sigma(x)|\nabla u|^2(x)dx : u \in W^{1,2}_{#,E}(Y; \mathbb{R}^2) \right\} \] (1.10)
Here \( E = e_1 \otimes \xi + e_2 \otimes \xi' \in \mathbb{R}^{2 \times 2} \), \( W^{1,2}_{#,E} \) is the vectorial counterpart of \( W^{1,2}_{#,\xi} \), and \( A \cdot B = \text{Tr} A^T B = \sum A_{ij} B_{ij} \). Having transformed the problem into a vectorial one permits the direct use of rigidity of gradient fields. It also displays in a very direct way the strong connection between bounds on conductivities and vectorial variational problems, including the concept of quasiconvexity. One crucial observation is that, the determinant being a divergence, one has
\[ \int_Y \det \nabla u \, dx = \det E \quad \text{for all} \ u \in W^{1,2}_{#,E}(Y; \mathbb{R}^2). \] (1.11)
This condition permits a straightforward proof of (1.3), see § 2; in the language of the calculus of variations it turns out that the Hashin-Shtrikman bound is the best polyconvex bound.

Gradient fields, however, satisfy many more constraints than (1.11). In particular, the improved bound obtained by Nesi and stated in (1.7) can be
obtained from the theorem of Alessandrini and Nesi (2001, Th. 1) which, in our case, shows that for any $u \in W^{1,2}_{\#,E}(Y; \mathbb{R}^2)$ which solves
\[ \text{div} (\sigma \nabla u) = 0 \text{ weakly in } Y \]
for some $\sigma \in \Sigma$, the sign of the determinant is prescribed by the boundary data in the sense that if $\det E > 0$ then $\det \nabla u > 0$ almost everywhere. A precise statement is given in Theorem 3.1 below. Originally, the bound was based on a weaker result established in Bauman et al. (2001).

The key idea in the proof of Theorem 1.1 is that for all $\sigma \in \Sigma$ such that the two sides of (1.3) are close, the gradient field $\nabla u$ entering (1.10) is close to the set (after scaling, see § 2 for details)
\[
Z = \{ F \in \mathbb{R}^{2 \times 2} : F = F^T, \text{Tr} F = 2 \} \cup \{ F \in \mathbb{R}^{2 \times 2} : F = \alpha \text{Id}, 0 < \alpha \leq 1 \}
\]
(1.12)
(a matrix $F \in \mathbb{R}^{2 \times 2}$ is anticonformal if it is traceless and symmetric). Using Alessandrini and Nesi’s result, we then show that $\nabla u$ is actually close to the smaller set
\[
Z^b = \{ F \in Z : \det F \geq 0 \}
\]
The same sets $Z$ and $Z^b$ were already used in proving (1.8) in Albin et al. (2007); one key ingredient of that proof was indeed that $\nabla u \in Z$ a.e. implies $\nabla u \in Z^b$ a.e.. Here we make the estimate quantitative. Precisely, we obtain the following optimal quantitative version of Alessandrini and Nesi’s estimate for this pair of sets.

**Theorem 1.2.** For all $E \in \mathbb{R}^{2 \times 2}$ with $\det E > 0$, and all $u \in W^{1,2}_{\#,E}(Y; \mathbb{R}^2)$, one has
\[
\| \text{dist}(\nabla u, Z^b) \|_{L^2} \leq c\| \text{dist}(\nabla u, Z) \|_{L^2}^{2/3} + c\| \text{dist}(\nabla u, Z) \|_{L^2},
\]
for some $c > 0$ independent of $E$ and $u$. The scaling of the estimate is optimal.

A similar result can be obtained for gradient Young measures supported on the set $Z$ (see § 3.3 for an illustration of the concept of gradient Young measure).

**Theorem 1.3.** Let $\nu$ be a $W^{1,p}$-gradient Young measure with average $E$ for some $p > 1$ and with $\text{supp} \nu \subset Z$, $\det E > 0$. Then $\nu$ is actually a $W^{1,\infty}$-gradient Young measure, and $\text{supp} \nu \subset Z^b$.

The study of rigidity results for gradient fields, such as Theorem 1.2, or for Young measures, such as Theorem 1.3, has been an important theme in the vectorial calculus of variations over many years. This ranges from classical results such as Korn’s inequality or Liouville’s rigidity theorem, to recent breakthroughs, including in particular the optimal quantitative version of Liouville’s rigidity derived by Friesecke, James and Müller (2002), corresponding two-well results (Chaudhuri & Müller 2004; De Lellis & Székelyhidi 2006), rigidity for the four gradient problem by Chlebík and Kirchheim.
(2002), rigidity of conformal matrices (Faraco & Zhong 2005), and the localization result for gradient Young measures by Faraco and Székelyhidi (2006). In all these cases one deals with gradient fields supported around a set with some specific structure (such as $Z$ here), and proves that the gradient constraint makes only a part of the set efficiently usable ($Z_b$ here).

The work by Faraco and Székelyhidi (2006) contains some results on sets which are very similar to the set $Z$ considered here. In particular, they show that if a gradient field or a $W^{1,2}$ gradient Young measure is supported on the set

$$Z_k = \left\{ F \in \mathbb{R}^{2\times 2} : F = F^T, \frac{1}{k} \leq \frac{2 - \lambda_1}{\lambda_2}, \frac{2 - \lambda_2}{\lambda_1} \leq k \right\},$$

for some $k > 0$, then it is supported either in its bounded or in its unbounded component (by taking the closure, the same extends to the limit $k \to \infty$).

Here $\lambda_1$ and $\lambda_2$ denote the singular values of $F$. Although the context is similar, their result and their method of proof differ significantly from the present work.

One common key ingredient in the proof of most of the cited rigidity results is that one separates the gradient field in two components: one which solves some “good” equation, which has to be constructed for the purpose, and one which is small. In Friesecke, James and Müller (2002) and generalizations the equation was Laplace’s equation, in Faraco and Székelyhidi (2006) it was a nonlinear Beltrami equation. In the present paper it is an almost degenerate linear elliptic equation, see (3.3) below, or equivalently, a linear Beltrami equation; the origin of the exponent $2/3$ in (1.13) is related to the degeneracy of the equation.

The proofs of these results are given in the following sections. In §2 we prove Theorem 1.1; in §3 we prove Theorem 1.2; and in §4 we prove Theorem 1.3.

2. The improved bound

Before proving the improved bound, we present a proof of the bound (1.3) using the techniques developed in (Tartar 1979; Lurie & Cherkaev 1984; Tartar 1985).

**Theorem 2.1** (The Hashin-Shtrikman Bound). Let $0 < \sigma_1 < \sigma_2 < \sigma_3$ and $m_1, m_2, m_3 \geq 0$ be given with $m_1 + m_2 + m_3 = 1$. Let $\sigma \in \Sigma$ (see (1.4)) be given such that $\sigma_{\text{hom}} = s_{\text{hom}} \text{Id}$. Then

$$s_{\text{hom}} \geq \left( \int_Y (\sigma + \sigma_1)^{-1} dx \right)^{-1} - \sigma_1 = \left( \sum_{i=1}^{3} \frac{m_i}{\sigma_i + \sigma_1} \right)^{-1} - \sigma_1. \quad (2.1)$$

**Proof.** We begin with the characterization of $\sigma_{\text{hom}}$ given in (1.10), and rewrite the integral using (1.11), as

$$\int_Y \sigma(x) |\nabla u|^2(x) = \int_Y \left( \sigma(x) |\nabla u|^2(x) + 2\sigma_1 \det \nabla u(x) \right) dx - 2\sigma_1 \det E.$$
The integrand on the right-hand side can be seen as a quadratic form in $\nabla u$. We denote it more compactly by

$$\sigma(x)|F|^2 + 2\sigma_1 \det F = F \cdot L(x)F$$

where $L(x) : \mathbb{R}^{2 \times 2} \to \mathbb{R}^{2 \times 2}$ is the position-dependent linear operator defined as

$$L(x)F = \sigma(x)F + \sigma_1 \text{cof } F.$$  
(2.2)

It is straightforward to check that this operator is self-adjoint and positive semi-definite almost everywhere. Furthermore, $L$ has a nontrivial null-space only for those $x$ for which $\sigma(x) = \sigma_1$ (see Remark 2.2). This notation gives an alternate form for the characterization of $\sigma_{\text{hom}}$. Namely,

$$E^T \cdot \sigma_{\text{hom}} E^T = \min \left\{ \int_Y \nabla u \cdot L \nabla u \, dx - 2\sigma_1 \det E : u \in W^{1,2}_{\#,E}(Y; \mathbb{R}^2) \right\}.$$  
(2.3)

The bound (2.1) arises by removing the constraint that the field $\nabla u$ is a gradient:

$$E^T \cdot \sigma_{\text{hom}} E^T \geq \min \left\{ \int_Y F \cdot LF \, dx - 2\sigma_1 \det E : F \in L^2(Y; \mathbb{R}^{2 \times 2}), \int_Y F \, dx = E \right\}.$$  
(2.4)

The inequality follows from the observation that for any $u \in W^{1,2}_{\#,E}(Y; \mathbb{R}^2)$, $\nabla u$ is an admissible $F$ in the latter minimization problem.

This minimization problem is straightforward to address. We seek to minimize a quadratic functional subject to a linear constraint. Fix $E = \frac{1}{\sqrt{2}} \text{Id}$. From the Euler-Lagrange equations of (2.4), it is easy to see that a minimizer $F$ of the problem satisfies $LF = G$ a.e. in $Y$ for some $G \in \mathbb{R}^{2 \times 2}$. In particular, if we take

$$F(x) = \frac{1}{\sqrt{2}(\sigma(x) + \sigma_1)} \left( \sum_{i=1}^{3} \frac{m_i}{\sigma_i + \sigma_1} \right)^{-1} \text{Id},$$  
(2.5)

then $F$ has the correct average and

$$LF = \frac{1}{\sqrt{2}} \left( \sum_{i=1}^{3} \frac{m_i}{\sigma_i + \sigma_1} \right)^{-1} \text{Id} \text{ a.e. in } Y.$$  
(2.6)

It follows easily that $F$ is a minimizer for (2.4). Substituting the definition of $E$, (2.5) and (2.6) into (2.4) gives (2.1).

\[ \Box \]

**Remark 2.2.** To understand $L$, it is useful to study the operator independently in the sets

$$Y_i = \{ x \in Y : \sigma(x) = \sigma_i \},$$
and to recall that for all $F \in \mathbb{R}^{2 \times 2}$ one has
\[
\text{cof} \left( \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \right) = \left( \begin{pmatrix} F_{22} & -F_{21} \\ -F_{12} & F_{11} \end{pmatrix} \right), \quad \text{cof}(\text{cof}(F)) = F.
\]

The operator $L$ has eigenvalues $\sigma_i + \sigma_1$ and $\sigma_i - \sigma_1$, with associated eigenspaces
\[
\mathbb{R}_+^{2 \times 2} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\} \quad \text{and} \quad \mathbb{R}_-^{2 \times 2} = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} : a, b \in \mathbb{R} \right\}
\]
respectively. In particular, $L$ is invertible in $Y_i$ for $i \neq 1$. In $Y_1$, $L$ has the non-trivial nullspace $\mathbb{R}_-^{2 \times 2}$. Its eigenvalues are $2\sigma_1$ and $0$ with the same eigenspaces as above. Thus, in $Y_i$, (2.6) uniquely determines only the projection of $F$ onto $\mathbb{R}_+^{2 \times 2}$.

We now turn our attention to the proof of Theorem 1.1. The idea is to get a quantitative estimate on the error in passing from (2.3) to (2.4).

**Proof of Theorem 1.1. Step 1. Estimate of the HS defect by a good comparison field.** In order to simplify some computations, we shall rescale the problem so that
\[
E = \sum_{i=1}^{3} \frac{m_i}{\sigma_i + \sigma_1} \text{Id}.
\]
Then the optimality condition (2.6) becomes
\[
LF = \text{Id} \quad \text{a.e. in } Y.
\]
(2.8)

We fix the volume fractions $m_1, m_2, m_3$ and let $\sigma \in \Sigma$ be an admissible material layout with $\sigma_{\text{hom}} = s_{\text{hom}} \text{Id}$. We now split any $u \in W^{1,2}(\mathbb{R}^2)$ into a solution to (2.8) and an error. Specifically, we define $F_u \in L^2(Y; \mathbb{R}^{2 \times 2})$ by
\[
F_u(x) = \begin{cases} \frac{1}{2\sigma_1} \sigma_1 + \sigma_i \text{Id} & \text{if } x \in Y_i, \ i \neq 1, \\ \frac{1}{2\sigma_1} \sigma_1 + \sigma_i \text{Id} + \frac{1}{2} (\nabla u(x) - \text{cof}(\nabla u(x))) & \text{if } x \in Y_1,
\end{cases}
\]
(2.9)
with $Y_i = \{ x \in Y : \sigma(x) = \sigma_i \}$. Thus $F_u$ is either a multiple of the identity or of the form $\beta \text{Id} + H$, with $H \in \mathbb{R}^{2 \times 2}$; we shall later rescale $F_u$ to a matrix field $G$ taking values in $Z$. The subscript $u$ makes explicit the dependence of $F_u$ on $u$.

Since, by construction, $F_u$ satisfies (2.8) almost everywhere, we have
\[
\int_Y (\nabla u - F_u) \cdot LF_u \, dx = \int_Y \text{Tr}(\nabla u - F_u) \, dx = 0.
\]
(2.10)
The last equality holds because
\[
\int_Y \text{Tr} F_u \, dx = 2 \sum_{i=1}^{3} \frac{m_i}{\sigma_i + \sigma_1} = \text{Tr} E = \int_Y \text{Tr} \nabla u \, dx.
\]
From (2.8), (2.9) and (2.10), we find
\[
\begin{align*}
\int_Y \nabla u \cdot L \nabla u \, dx &= \int_Y (\nabla u - F_u) \cdot L(\nabla u - F_u) \, dx \\
&= \int_Y F_u \cdot L F_u + \int_Y (\nabla u - F_u) \cdot L(\nabla u - F_u) \, dx \\
&= 2 \sum_{i=1}^{3} \frac{m_i}{\sigma_i + \sigma_1} + \int_Y (\nabla u - F_u) \cdot L(\nabla u - F_u) \, dx.
\end{align*}
\]
This, together with the characterization (2.3) of \( \sigma_{\text{hom}} = s_{\text{hom}} \text{Id} \) shows that
\[
s_{\text{hom}} = s_{\text{HS}} + E,
\]
where
\[
E = \frac{1}{2} \left( \sum_{i=1}^{3} \frac{m_i}{\sigma_i + \sigma_1} \right)^{-2} \min \left\{ \int_Y (\nabla u - F_u) \cdot L(\nabla u - F_u) \, dx : u \in W^{1,2}_{\# E}(Y; \mathbb{R}^2) \right\}.
\]
The improved bound arises by proving a lower bound on \( E \).

In \( Y_i \) for \( i \neq 1 \), we have (see Remark 2.2)
\[
(\nabla u - F_u) \cdot L(\nabla u - F_u) \geq (\sigma_i - \sigma_1)|\nabla u - F_u|^2.
\]
In \( Y_1 \) instead we have \( \nabla u - F_u \in \mathbb{R}^{2 \times 2}_+ \), and from Remark 2.2 we see that
\[
L(\nabla u - F_u) = 2\sigma_1(\nabla u - F_u).
\]
From these two observations, we find that
\[
\int_Y (\nabla u - F_u) \cdot L(\nabla u - F_u) \geq 2\sigma_1 \int_{Y_1} |\nabla u - F_u|^2 + \sum_{i=2}^{3} (\sigma_i - \sigma_1) \int_{Y_i} |\nabla u - F_u|^2.
\]
Thus, we have
\[
E \geq C \inf \left\{ \| \nabla u - F_u \|_{L^2}^2 : u \in W^{1,2}_{\# E}(Y; \mathbb{R}^2) \right\},
\]
where
\[
C = 2\sigma_1^2 \min\{2\sigma_1, \sigma_2 - \sigma_1\} \leq \frac{1}{2} \left( \sum_{i=1}^{3} \frac{m_i}{\sigma_i + \sigma_1} \right)^{-2} \min\{2\sigma_1, \sigma_2 - \sigma_1\}.
\]
Indeed, (2.12) holds if \( C \) is replaced by the larger term in the above inequality. We have chosen \( C \) independent of the volume fractions \( m_i \).

Step 2. Lower bound for \( \| \nabla u - F_u \|_{L^2}^2 \) using the rigidity estimate. It is convenient to make another affine change of variables to bring the set \( Z \) into play. We set
\[
v(x) = \gamma \left( u(x) - \frac{1}{\sigma_3 + \sigma_1} x \right) \quad \text{and} \quad G(x) = \gamma \left( F_u(x) - \frac{1}{\sigma_3 + \sigma_1} \text{Id} \right),
\]
where
\[
\gamma = \frac{2\sigma_1(\sigma_3 + \sigma_1)}{\sigma_3 - \sigma_1}, \quad (2.13)
\]
Then $\nabla v - G = \gamma(\nabla u - F_u)$, and

$$G(x) = \begin{cases} \alpha_i \text{Id} & \text{if } x \in Y_i, \ i \neq 1, \\ \alpha_1 \text{Id} + \frac{1}{2}(\nabla v(x) - \text{cof}(\nabla v(x))) & \text{if } x \in Y_1, \end{cases}$$

(2.14)

where

$$\alpha_i = \frac{2\sigma_1(\sigma_3 - \sigma_i)}{(\sigma_i + \sigma_1)(\sigma_3 - \sigma_1)} \quad \text{for } i = 1, 2, 3.$$  

That is, $\alpha_1 = 1$, $\alpha_3 = 0$, and $\alpha_2 = \alpha$ from (1.5). Thus, $G \in \tilde{Z} = Z \cup \{0\}$ almost everywhere, with $Z$ as in (1.12). Further, $v \in W^{1,2}(Y; \mathbb{R}^2)$, where

$$\tilde{E} = \sum \alpha_i m_i \text{Id} = (m_1 + \alpha m_2)\text{Id}.$$  

In particular, $\det \tilde{E} > 0$.

We have to establish a lower bound for $\|\nabla v - G\|_{L^2}^2$ if (1.6) fails. Our proof is based on comparing the global relation (corresponding to (1.11))

$$\int_Y \det \nabla v dx = \det \tilde{E} = (m_1 + \alpha m_2)^2$$

(2.15)

with local estimates on $\det \nabla v$. As a warm-up we first consider the situation $\nabla v - G = 0 \ a.e.$ Then $\nabla v \in \tilde{Z}$ and by Theorem 1.2 $\det \nabla v \geq 0 \ a.e.$ Since $\nabla v = G = 0$ in $Y_3$ we see that

$$\int_Y \det \nabla v dx \geq \alpha^2 m_2.$$  

Comparing with (2.15) shows that $m_1 + \alpha m_2 \geq \alpha m_2^{1/2}$, which is equivalent to (1.6), concluding the proof in this warm-up case.

We now use the quantitative rigidity estimate to derive a lower bound for $\|\nabla v - G\|_{L^2}^2$ if (1.6) fails. We may without loss of generality only consider those $v$ for which $\|\nabla v - G\|_{L^2} \leq 1$ (otherwise $\|\nabla u - F_u\|_{L^2} \geq \gamma^{-1}$, and we may choose $\Delta = \gamma^{-2}C > 0$ in (1.9)). We start from $Y_i$, for $i \neq 1$, and write

$$\int_{Y_i} \det \nabla v \, dx = \int_{Y_i} \det(\nabla v - G + G) \, dx$$

$$= \int_{Y_i} \left[ \det(\nabla v - G) + (\nabla v - G) \cdot \text{cof} G + \det G \right] \, dx.$$  

Since $G = \alpha_i \text{Id}$ in $Y_i$, applying Hölder’s inequality to the second term we find

$$\int_{Y_i} \det \nabla v \, dx \geq \frac{1}{2} \int_{Y_i} \left| \nabla v - G \right|^2 \, dx - \sqrt{2}\alpha_i m_i^{1/2} \left( \int_{Y_i} \left| \nabla v - G \right|^2 \, dx \right)^{1/2} + m_i \alpha_i^2;$$

with two terms disappearing for $i = 3$ since $\alpha_3 = 0$. Equivalently,

$$\frac{\gamma^2}{2} \|\nabla u - F_u\|_{L^2(Y_i)}^2 + \gamma \sqrt{2}\alpha_i m_i^{\frac{1}{2}} \|\nabla u - F_u\|_{L^2(Y_i)} \geq m_i \alpha_i^2 - \int_{Y_i} \det \nabla v \, dx.$$  

(2.16)

For $Y_1$ we proceed differently. Consider any mapping $H \in L^\infty(Y; \mathbb{R}^b)$. Then we estimate as before with $H$ in place of $G$, noting that $|H| \leq 2$ and
det \( H \geq 0 \) a.e.,
\[
\int_{Y_1} \det \nabla v \, dx = \int_{Y_1} [\det(\nabla v - H) + (\nabla v - H) \cdot \text{cof} \, H + \det H] \, dx \\
\geq -\frac{1}{2} \int_{Y_1} |\nabla v - H|^2 \, dx - 2m^\frac{1}{2}_1 \left( \int_{Y_1} |\nabla v - H|^2 \, dx \right)^{\frac{1}{2}} \\
\geq -\frac{1}{2} \int_Y |\nabla v - H|^2 \, dx - 2m^\frac{1}{2}_1 \left( \int_Y |\nabla v - H|^2 \, dx \right)^{\frac{1}{2}}.
\]

Taking the supremum in \( H \) of the right-hand side, we obtain
\[
\int_{Y_1} \det \nabla v \, dx \geq -\frac{1}{2} \| \text{dist}(\nabla v, Z^b) \|_L^2 - 2m^\frac{1}{2}_1 \| \text{dist}(\nabla v, Z^b) \|_L^2.
\]

After applying Theorem 1.2, we have
\[
\int_{Y_1} \det \nabla v \, dx \geq -\frac{1}{2} c \| \text{dist}(\nabla v, Z) \|_L^{\frac{4}{3}} - 2cm^\frac{1}{2}_1 \| \text{dist}(\nabla v, Z) \|_L^2,
\]
which translates into
\[
\frac{1}{2} c^\gamma \| \nabla u - F_u \|_{L^2(Y)} + 2cm^\frac{1}{2}_1 \gamma^\frac{2}{3} \| \nabla u - F_u \|_{L^2(Y)} \geq -\int_{Y_1} \det \nabla v \, dx, \tag{2.17}
\]
for a universal constant \( c > 0 \). Combining (2.15), (2.16) and (2.17), we find
\[
\frac{\alpha^2}{2} \| \nabla u - F_u \|_{L^2(Y \setminus Y_1)}^2 + \sqrt{2} \alpha m^\frac{1}{2}_1 \| \nabla u - F_u \|_{L^2(Y_2)}^2 \\
+ \frac{1}{2} c^\gamma \| \nabla u - F_u \|_{L^2(Y)}^2 + 2cm^\frac{1}{2}_1 \gamma^\frac{2}{3} \| \nabla u - F_u \|_{L^2(Y)}^2 \geq \kappa, \tag{2.18}
\]
where
\[
\kappa = \max \left\{ 0, \ m_2 \alpha^2 - (m_1 + m_2 \alpha)^2 \right\}, \tag{2.19}
\]
clearly \( \kappa > 0 \) when (1.6) is violated. We may take the maximum of the second term above with 0 since the left-hand side of (2.18) is clearly non-negative.

Using straightforward estimates on the left-hand side of (2.18), we find
\[
\| \nabla u - F_u \|_{L^2}^2 + \| \nabla u - F_u \|_{L^2}^2 + \| \nabla u - F_u \|_{L^2}^2 + \| \nabla u - F_u \|_{L^2}^2 \geq c^* \kappa,
\]
where \( c^* \) is a constant that depends on \( c \) and the \( \sigma_i \). Thus we have (see the comments before (2.17)) either \( \mathcal{E} \) is sufficiently large already, or else the above inequality holds, which implies that at least one of the terms on the left-hand side is larger than \( \kappa/4 \). Thus, by choosing a constant \( C^* > 0 \) sufficiently small and recalling (2.12), we have
\[
\mathcal{E} \geq C^* \min \left\{ 1, \kappa^3 \right\}
\]
and the right-hand side can be taken as \( \Delta \) for the theorem; \( C \) depends only on \( c \) and the \( \sigma_i \), but not on the \( m_i \). □

By more carefully estimating (2.11) and (2.18), we obtain the following refinement of the theorem.
Proposition 2.3. Theorem 1.1 holds with
\[
\Delta = \frac{1}{2\gamma^2} \left( \sum_{i=1}^{3} \frac{m_i}{\sigma_i + \sigma_1} \right)^{-2} \min\{2\sigma_1, \sigma_2 - \sigma_1\}
\times \min \left\{ 1, \frac{\kappa}{2}, \left( \frac{\kappa}{4\alpha} \right)^2 \frac{1}{2m_2}, \left( \frac{\kappa}{2\kappa} \right)^{\frac{3}{2}}, \left( \frac{\kappa}{8c} \right)^{\frac{3}{2}} \frac{1}{m_1^{\frac{3}{2}}} \right\}
\]
where \(\alpha\) is defined in (1.5), \(\gamma\) is defined in (2.13), \(\kappa\) is defined in (2.19), and \(c\) is universal.

3. Rigidity of the set \(Z\)

3.1. The underlying PDE. In order to illustrate the main ideas in the proof we first present a special case. Assume that \(u \in W^{1,2}_{\#,E}(Y;\mathbb{R}^2)\) satisfies the differential inclusion
\[
\nabla u \in Z_0 \text{ a.e.,}
\]
where
\[
Z_0 = \{ F \in \mathbb{R}^{2\times 2} : F = F^T, \text{ Tr } F = 2 \} \cup \left\{ \frac{1}{2} \text{Id} \right\},
\]
and assume that \(\det E > 0\). The key property of \(Z_0\) is that for any \(F \in Z_0\) there is a number \(\sigma_F > 0\) such that
\[
\sigma_F F + \text{cof } F = 2\text{Id}.
\]
Indeed, it suffices to take \(\sigma_F = 3\) if \(F = \text{Id}/2\), and \(\sigma_F = 1\) else. Performing this construction at each point \(x \in Y\) we obtain a function \(\sigma \in L^\infty(Y;\mathbb{R})\) such that
\[
\sigma \nabla u + \text{cof } \nabla u = 2\text{Id} \quad \text{a.e. in } Y
\]
(it suffices to take \(\sigma(x) = 3\) if \(\nabla u(x) = \frac{1}{2}\text{Id}\), and \(\sigma(x) = 1\) otherwise). Since the cofactor of a gradient field is divergence-free, taking the divergence (in a distributional sense) of (3.2) shows that \(u\) solves the elliptic equation
\[
\text{div} (\sigma \nabla u) = 0.
\]
Since \(1 \leq \sigma \leq 3\) this equation has a unique solution in \(W^{1,2}_{\#,E}(Y;\mathbb{R}^2)\), hence it uniquely identifies \(u\). This permits us to use the following injectivity result.

Theorem 3.1 (From Alessandrini & Nesi (2001), Thm. 1). Let \(u \in W^{1,2}_{\#,E}(Y;\mathbb{R}^2)\) be the weak solution of \(\text{div} (\sigma \nabla u) = 0\), where \(\sigma \in L^\infty(Y;\mathbb{R}^{2\times 2})\), \(\sigma = \sigma^T\), \(1/k \leq \sigma \leq k\) a.e., for some \(k \geq 1\), and \(E \in \mathbb{R}^{2\times 2}\). If \(\det E > 0\) then
\[
\det \nabla u > 0 \text{ a.e. in } Y.
\]

By (3.3), we can apply this result to our \(u\). Then, (3.1) becomes
\[
\nabla u \in Z_0 \cap \{ F : \det F > 0 \} \text{ a.e.}
\]
In particular, we have shown that any \(u \in W^{1,2}_{\#,E}(Y;\mathbb{R}^2)\) which solves the differential inclusion \(\nabla u \in Z_0\), where \(Z_0\) is the unbounded set defined above,
is actually Lipschitz, i.e., that only a bounded part of $Z_0$ is actually used. The key argument used to derive (3.4) has been to construct an elliptic PDE which has the given $u$ as a solution, and then to use properties of that PDE. In §33.2 we shall consider maps $u \in W^{1,2}_{#,E}(Y; \mathbb{R}^2)$ which are only approximate solutions of the differential inclusion, and derive an optimal quantitative rigidity for them. In §4 we shall instead consider maps with weaker integrability, and in particular $W^{1,p}$-Gradient Young Measures, and show that they exhibit the same rigidity. In §33.4 we show that our rigidity result cannot follow from polyconvexity alone.

3.2. Quantitative estimate.

**Proof of Theorem 1.2.** Let $F : Y \to Z$ be a measurable map such that $\text{dist}(\nabla u, Z) = |\nabla u - F|$. Pick some $\delta \in (0, 1)$. We define

$$F_\delta = (1 - \delta)F + \delta \text{Id}, \quad u_\delta(x) = (1 - \delta)u(x) + \delta x.$$ 

Clearly $F_\delta \in Z$; if $F_\delta$ is of the form $\alpha \text{Id}$, then $\delta \leq \alpha \leq 1$. At the same time $|F_\delta - \nabla u_\delta| = (1 - \delta)|F - \nabla u|$. We define $\sigma_\delta : Y \to \mathbb{R}$ by setting

$$\sigma_\delta(x) = \begin{cases} (2 - \alpha)/\alpha & \text{if } F_\delta(x) = \alpha \text{Id} \\ 1 & \text{otherwise}. \end{cases}$$

As above, this choice ensures that

$$\sigma_\delta F_\delta + \text{cof } F_\delta = 2 \text{Id} \text{ a.e. in } Y. \quad (3.5)$$

Now let

$$f_\delta = \sigma_\delta \nabla u_\delta + \text{cof } \nabla u_\delta - 2 \text{Id}. \quad (3.6)$$

It is clear that $f_\delta \in L^2(Y; \mathbb{R}^{2 \times 2})$, and that

$$\text{div}(\sigma_\delta \nabla u_\delta) = \text{div } f_\delta. \quad (3.7)$$

Given $f_\delta$, the elliptic equation (3.7) has a unique solution in $W^{1,2}_{#,E}(Y; \mathbb{R}^2)$, and this is $u_\delta$. Here and below $E_\delta = (1 - \delta)E + \delta \text{Id}$. We remark that if $\text{Tr } E \geq 0$ then $\det E_\delta > 0$ for all $\delta$, otherwise $\det E_\delta > 0$ provided $(1 - \delta) \text{Tr } E + \delta > 0$.

We define $v \in W^{1,2}_{#,E}(Y; \mathbb{R}^{2 \times 2})$ as the solution of

$$\text{div}(\sigma_\delta \nabla v_\delta) = 0.$$ 

By Theorem 3.1 we have $\det \nabla v_\delta > 0$ a.e. in $Y$, which implies

$$\text{dist}(\nabla v_\delta, Z^b) \leq \sqrt{2} \text{dist}(\nabla v_\delta, Z). \quad (3.8)$$

To see this, let $G \in \mathbb{R}^{2 \times 2}$ be such that $\text{dist}(G, Z) = |G - H|$ for some $H \in Z \setminus Z^b$ (otherwise there is nothing to prove). We can without loss of generality assume $G = \text{diag}(\lambda_1, \lambda_2)$, with $\lambda_2 > 2 + \lambda_1 > 2$ (consider the plane of diagonal matrices, where $\bar{Z} \setminus Z^b$ is a subset of a straight line). We compute $|G - H| = (\lambda_1 + \lambda_2 - 2)/\sqrt{2}$, and $\text{dist}(G, Z^b) \leq \text{dist}(G, \text{diag}(0, 2)) = (\lambda_1^2 + (\lambda_2 - 2)^2)^{1/2} \leq \lambda_1 + \lambda_2 - 2$. 


It remains to estimate the distance between $\nabla u_\delta$ and $\nabla v_\delta$. To do this, we consider the difference $w_\delta = u_\delta - v_\delta$. By the linearity of the equation, $w_\delta$ is the unique solution in $W^{1,2}_{\#; 0}(Y; \mathbb{R}^2)$ of

$$\text{div}(\sigma_\delta \nabla w_\delta) = \text{div} f_\delta.$$  

Testing this equation with $w_\delta$ gives

$$\int_Y \sigma_\delta |\nabla w_\delta|^2 dx = \int_Y f_\delta \cdot \nabla w_\delta dx.$$  

Let $p_\delta = \sigma_\delta^{1/2} |\nabla w_\delta|$. Then $\|p_\delta\|_{L^2}^2 \leq \|p_\delta\|_{L^2} \|\sigma_\delta^{-1/2} f_\delta\|_{L^2}$, and thus

$$\|p_\delta\|_{L^2(Y)} \leq \|\sigma_\delta^{-1/2} f_\delta\|_{L^2(Y; \mathbb{R}^2)}.$$  

(3.9)

In order to estimate the norm of $\sigma_\delta^{-1/2} f_\delta$, we combine (3.5) and (3.6) to obtain

$$f_\delta = \sigma_\delta (\nabla u_\delta - F_\delta) + (\text{cof} \nabla u_\delta - \text{cof} F_\delta)$$  

which implies that pointwise

$$\sigma_\delta^{-1/2} |f_\delta| \leq (\sigma_\delta^{1/2} + \sigma_\delta^{-1/2}) |\nabla u_\delta - F_\delta| \leq (\sigma_\delta^{1/2} + \sigma_\delta^{-1/2}) \text{dist}(\nabla u, Z).$$  

(3.10)

Therefore, by (3.9) and the fact that by definition $1 \leq \sigma_\delta \leq 2/\delta$, we obtain

$$\|\nabla w_\delta\|_{L^2} \leq \|\sigma_\delta^{-1/2}\|_{L^\infty} \|p_\delta\|_{L^2} \leq \|p_\delta\|_{L^2} \leq \frac{c}{\delta^{1/2}} \|\text{dist}(\nabla u, Z)\|_{L^2}.$$  

Finally, recalling (3.8),

$$\|\text{dist}(\nabla u_\delta, Z^b)\|_{L^2} \leq \|\text{dist}(\nabla v_\delta, Z^b)\|_{L^2} + \|\nabla u_\delta - \nabla v_\delta\|_{L^2} \leq \sqrt{2} \|\text{dist}(\nabla v_\delta, Z)\|_{L^2} + \|\nabla w_\delta\|_{L^2} \leq \sqrt{2} \|\text{dist}(\nabla u_\delta, Z)\|_{L^2} + (1 + \sqrt{2}) \|\nabla w_\delta\|_{L^2} \leq \sqrt{2} \|\text{dist}(\nabla u, Z)\|_{L^2} + (1 + \sqrt{2}) \|\nabla w_\delta\|_{L^2},$$

and, for all admissible $\delta$,

$$\|\text{dist}(\nabla u, Z^b)\|_{L^2} \leq \frac{\sqrt{2} \delta}{1 - \delta} + \frac{1}{1 - \delta} \|\text{dist}(\nabla u_\delta, Z^b)\|_{L^2} \leq c \delta + c \|\text{dist}(\nabla u, Z)\|_{L^2} + \frac{c}{\delta^{1/2}} \|\text{dist}(\nabla u, Z)\|_{L^2}.$$  

Careful optimization in $\delta$ (when $\text{Tr } E < 0$, choose $\delta \leq |\text{Tr } E| \leq \sqrt{2} \|\text{dist}(\nabla u, Z)\|_{L^1} \leq \sqrt{2} \|\text{dist}(\nabla u, Z)\|_{L^2}$) we conclude

$$\|\text{dist}(\nabla u, Z^b)\|_{L^2} \leq c \|\text{dist}(\nabla u, Z)\|_{L^2}^{2/3} + c \|\text{dist}(\nabla u, Z)\|_{L^2}.$$  

Optimality of the scaling follows from Lemma 3.2 below. □
3.3. Optimality: gradient Young measures and laminates. Optimality is proven constructing a gradient field such that its $L^2$ distance from $Z$ is of order $\varepsilon$, and its $L^2$ distance from $Z^b$ is of order $\varepsilon^{2/3}$. Instead of writing an explicit test function, it is much simpler to first perform a construction in matrix space, and then to use general tools, in particular the concept of a laminate, to obtain existence of a test function. We therefore start by briefly sketching the necessary background material.

Given a sequence $(u_k)_{k \in \mathbb{N}}$ converging weakly in $W^{1,p}_{#;E}(Y;\mathbb{R}^2)$ to the affine function $x \mapsto Ex$, one says that the sequence $(u_k)$ generates the $W^{1,p}$ (homogeneous) gradient Young measure $\nu \in \mathcal{M}(\mathbb{R}^{2\times 2})$ if

$$
\lim_{k \to \infty} \int_Y f(\nabla u_k) \, dx = \int_{\mathbb{R}^{2\times 2}} f(F) \, d\nu(F) \tag{3.11}
$$

for all $f \in C^0_c(\mathbb{R}^{2\times 2})$. By simple truncation arguments the same will automatically hold for all continuous $f$ satisfying for some $q < p$ the growth condition $|f(F)| \leq C(1 + |F|^q)$. Taking $f$ to be the identity mapping one sees that the average of $\nu$ is $E$. It can be shown that every weakly converging sequence generates such a measure. The Young measure gives the volume distribution of the values of the gradient, see, e.g., Pedregal (1997), Müller (1999) for details.

It is a very difficult question to decide which measures on $\mathbb{R}^{2\times 2}$ can be generated this way. There is, however, a large class of measures that can be easily generated, the class of so-called laminates. A laminate of zeroth order with average $F$ is a Dirac delta, i.e., it has the form $\nu = \delta_F$, and is generated by the (constant) sequence $u_k(x) = Fx$. A laminate of $n$-th order is defined inductively from a laminate of order $n-1$ by replacing each of the terms $c\delta_F$ by a sum $c(\lambda \delta_{F_1} + (1-\lambda) \delta_{F_2})$, where $\lambda F_1 + (1-\lambda) F_2 = F$, $\lambda \in [0,1]$, and $\text{rank}(F_1 - F_2) \leq 1$. One says that the matrix $F$ has been split into $F_1$ and $F_2$. For example, first-order laminates have the form $\nu = \lambda \delta_{F_1} + (1-\lambda) \delta_{F_2}$, with $F_1 - F_2 = a \otimes n$, and are the limits of the gradient distributions of the maps

$$u_k(x) = F_2 x + \frac{1}{k} a \chi(kx \cdot n).$$

The Lipschitz function $\chi : \mathbb{R} \to \mathbb{R}$ is defined by $\chi(0) = 0$, $\chi'(t) = 1$ if $t \in (z, z + \lambda)$ and $\chi'(t) = 0$ if $t \in (z + \lambda, z + 1)$, for $z \in \mathbb{Z}$. For large $k$, the gradients $\nabla u_k$ oscillate on a fine scale between the values $F_1$ and $F_2$, with average $F = \lambda F_1 + (1-\lambda) F_2$. As $k \to \infty$, the sequence $u_k$ converges weakly-*$ in $W^{1,\infty}$ to the affine function $x \mapsto Fx$; the function $f(\nabla u_k)$ converges weakly-* in $L^\infty$ to $\int f(F) \, d\nu(F) = \lambda f(F_1) + (1-\lambda) f(F_2)$. Refining this argument one can show that mixtures are always possible between rank-one connected matrices, hence that all laminates as defined above are attainable as weak limits of gradients. For details see, e.g., Dacorogna (1989), Müller (1999), Dolzmann (2003).
**Lemma 3.2.** For any $\epsilon > 0$ there exists $E \in \mathbb{R}^{2\times 2}$ with $\det E > 0$ and $u \in W^{1,2}_{\text{#},E}(Y; \mathbb{R}^2)$ such that

$$\|\text{dist}(\nabla u, Z)\|_{L^2} \leq \epsilon$$

and

$$\|\text{dist}(\nabla u, Z^b)\|_{L^2} \geq \frac{1}{4}\epsilon^{2/3}.$$  

**Proof.** If $\epsilon \geq 1$ one can take $u(x) = Ex = (1 + \epsilon/2)x$, therefore we can assume $\epsilon < 1$. We shall first construct a laminate which obeys the mentioned inequalities, and then a test function $u$. The laminate is supported on diagonal matrices. Let $0 < \delta < 1$. We consider $F_1 = \text{diag}(\delta, 2 - \delta) \in Z^b$; $E = \text{diag}(\delta/2, 2 - \delta)$; $F' = (-\delta, 2 - \delta)$. Then

$$\nu' = \frac{3}{4}\delta F_1 + \frac{1}{4}\delta F'$$

is a laminate with average $E$. We now split $F'$ into the two matrices $F_2 = \text{diag}(-\delta, 2 + \delta)$ and $F_3 = \text{diag}(-\delta, 0)$, to obtain the laminate

$$\nu = \frac{3}{4}\delta F_1 + \frac{1}{2}\delta F_2 + \frac{1}{4}\delta F_3$$

(see Figure 1). Since $F_1, F_2 \in Z$, we have

$$\int_{\mathbb{R}^{2\times 2}} \text{dist}^2(F, Z) d\nu(F) = \frac{1}{4} \frac{2\delta}{42 + \delta} \text{dist}^2(F_3, Z) = \frac{1}{4} \frac{2\delta}{42 + \delta} \delta^2 \leq \frac{1}{4} \delta^3.$$  

However, $F_2 \notin Z^b$. Therefore

$$\int_{\mathbb{R}^{2\times 2}} \text{dist}^2(F, Z^b) d\nu(F) = \frac{1}{4} \frac{2\delta}{42 + \delta} \text{dist}^2(F_2, Z^b) + \frac{1}{4} \frac{2\delta}{42 + \delta} \text{dist}^2(F_3, Z^b)$$

$$= \frac{1}{4} \frac{2\delta}{42 + \delta} 2\delta^2 + \frac{1}{4} \frac{2\delta}{42 + \delta} \delta^2 \geq \frac{1}{4} \delta^2.$$  

**Figure 1.** Sketch of the laminate used in the proof of Lemma 3.2.
Let now \( \delta = \varepsilon^{2/3} \), and pick a sequence \( u_k \) which generates the laminate \( \nu \). Then
\[
\lim_{k \to \infty} \int_Y \text{dist}^2(\nabla u_k, Z^b) dx \geq \frac{1}{4} \varepsilon^{4/3}, \quad \lim_{k \to \infty} \int_Y \text{dist}^2(\nabla u_k, Z) dx \leq \frac{1}{4} \varepsilon^2.
\]
Therefore taking \( k \) sufficiently large the Lemma is proven. \( \square \)

3.4. **Rigidity does not hold for polyconvex measures.** We finally show that our rigidity result uses in a substantial way the fact that we are dealing with a gradient field. In particular, the same cannot be proven just arguing with null Lagrangians, i.e., on the basis of polyconvex bounds. To prove this, we exhibit a polyconvex measure which violates the statement. By polyconvex measure we mean a measure \( \nu \) on \( \mathbb{R}^{2 \times 2} \) which obeys the equivalent for measures of (1.11), i.e., such that the determinant of the average is the average of the determinant. In particular, let \( E = \text{diag}(\frac{1}{7}, \frac{1}{3}) \), and
\[
\nu = \frac{16}{21} \delta_{\text{diag}(0,0)} + \frac{4}{21} \delta_{\text{diag}(1,1)} + \frac{1}{21} \delta_{\text{diag}(-1,3)}.
\]
It is clear that \( \text{supp} \nu \subset Z \), and that \( \text{det} E > 0 \). Easy computations prove that
\[
\langle \nu; \text{Id} \rangle = E, \quad \langle \nu; \text{det} \rangle = \text{det} E,
\]
hence \( \nu \) is a polyconvex measure with average \( E \). However, it is apparent that \( \text{supp} \nu \not\subset Z^b \).

4. **Gradient Young measures supported on \( Z \)**

The key point in the proof of Theorem 1.3 is that a \( W^{1,p} \)-gradient Young measure supported on \( Z \) necessarily has much better integrability, precisely, it is a \( W^{1,q} \)-gradient Young measure for any \( q \geq 1 \). Let us suppose that \( \nu \) is a \( W^{1,q} \)-gradient Young measure for some \( q > 2 \) and which is generated by a sequence \( \{u_j\} \). Taking \( f(F) = \text{dist}^2(F, Z) \) in (3.11) gives \( \| \text{dist}(\nabla u_j, Z) \|_{L^2} \to 0 \). Theorem 1.2 implies that \( \| \text{dist}(\nabla u_j, Z^b) \|_{L^2} \to 0 \) as well. Finally, applying (3.11) now for \( f(F) = \text{dist}^2(F, Z^b) \) shows that \( \text{supp} \nu \subset Z^b \). Moreover, the boundedness of \( \text{supp} \nu \) then implies by a truncation argument that \( \nu \) is in fact a \( W^{1,\infty} \)-gradient Young measure.

The proof of sufficient integrability relies on the following theorem due to Boyarskiï (1957) (see also Meyers (1963) for the generalization in \( n \) dimensions).

**Theorem 4.1** (From Boyarskiï (1957), Meyers (1963)). Let \( k \geq 1 \) be given. There exist constants \( p_c(k) < 2 < q_c(k) \) such that if \( p_c(k) < p < q_c(k) \), if \( \sigma = \sigma^T \in L^\infty(\mathcal{Y} ; \mathbb{R}^{2 \times 2}) \) satisfies
\[
\forall \xi \in \mathbb{R}^2 \quad k^{-1} |\xi|^2 \leq \xi \cdot \sigma \xi \leq k |\xi|^2 \text{ a.e.} \tag{4.1}
\]
and if \( f \in L^p(\mathcal{Y} ; \mathbb{R}^{2 \times 2}) \), then a unique solution to the equation
\[
\begin{aligned}
\text{div}(\sigma \nabla u) &= \text{div} f \\
u &\in W^{1,p}_0(\mathcal{Y} ; \mathbb{R}^2)
\end{aligned}
\]
exists and \( \|u\|_{W^{1,p}} \leq C(k,p)\|f\|_{L^p} \). The exponents \( p_c(k) \) and \( q_c(k) \) obey
\[
\lim_{k \to 1^+} p_c(k) = 1, \quad \lim_{k \to 1^+} q_c(k) = +\infty.
\]

Theorem 1.3 now follows from the following higher integrability result, which for notational simplicity is formulated in \( W^{1,3} \) (any exponent \( p > 2 \) would do).

**Lemma 4.2.** Let \( \nu \) be a homogeneous \( W^{1,p}\)-gradient Young measure with \( \text{supp} \nu \subset Z \) and \( p > 1 \). Then \( \nu \) is also a \( W^{1,3}\)-gradient Young measure.

**Proof.** By our initial discussion, we may assume \( p \leq 2 \). We begin by choosing \( k > 1 \) such that \( p_c(k) < p < 3 < q_c(k) \) where \( p_c(k) \) and \( q_c(k) \) are defined in Theorem 4.1. Let \( \{u^j\} \subset W^{1,p} \) be a generating sequence for \( \nu \). By a standard localization argument, we may assume
\[
\delta x \mapsto u^j(x) - Ex \in W^{1,p}_0(Y;\mathbb{R}^2) \tag{4.2}
\]
where \( E = \langle \nu; \text{Id} \rangle \) is the average of \( \nu \).

Now we proceed analogously to the proof of Theorem 1.2, with more care to the growth. Let \( q \) be such that \( p_c(k) < q < p \), and let \( \delta \) be such that
\[
\frac{2}{k+1} < \delta < 1.
\]
This ensures that the \( \sigma_\delta \) we shall construct will satisfy (4.1). For each \( j \) we define \( u^j_\delta, F^j_\delta, \sigma^j_\delta \) and \( f^j_\delta \) as in the proof of Theorem 1.2. Then (3.10) gives
\[
|f^j_\delta| = |\sigma^j_\delta \nabla u^j_\delta + \text{cof} \nabla u^j_\delta - 2 \text{Id}| \leq C_\delta \text{dist}(\nabla u^j, Z) \text{ a.e. in } Y,
\]
and since \( q < p \), (3.11) shows
\[
\lim_{j \to \infty} \|f^j_\delta\|_{L^q} \leq C_\delta \lim_{j \to \infty} \int \text{dist}^q(\nabla u^j, Z)dx = C_\delta \int_{\mathbb{R}^2} \text{dist}^q(F, Z)d\nu(F) = 0.
\]

Now consider the two PDEs
\[
\begin{cases}
\text{div}(\sigma^j_\delta \nabla(u^j_\delta - v^j_\delta)) = \text{div} f^j_\delta \text{ in } Y, \\
u^j_\delta - v^j_\delta \in W^{1,q}_0(Y;\mathbb{R}^2)
\end{cases} \tag{4.3}
\]
and
\[
\begin{cases}
\text{div}(\sigma^j_\delta \nabla v^j_\delta) = 0 \text{ in } Y, \\
x \mapsto v^j_\delta(x) - Ex \in W^{1,3}_0(Y;\mathbb{R}^2).
\end{cases} \tag{4.4}
\]
Since, by construction, \( p_c(k) < q < 3 < q_c(k) \), Theorem 4.1 guarantees the existence of a unique solution to both equations. Furthermore, since \( \text{div}(\sigma^j_\delta \nabla u^j_\delta) = \text{div} f^j_\delta \) and recalling (4.2), the solutions to (4.3) and (4.4) necessarily coincide. The estimate in Theorem 4.1 applied to (4.4) implies that \( \|v^j_\delta\|_{W^{1,3}} \leq C \|\sigma^j_\delta E\|_{L^3} \leq C k \|E\| \), hence the sequence \( \{v^j_\delta(x) - \delta x\} \) is bounded in \( W^{1,3} \) and it generates a \( W^{1,3}\)-gradient Young measure. Moreover, by Theorem 4.1 applied to (4.3), we have
\[
\|u^j_\delta - v^j_\delta\|_{W^{1,q}} \leq C(k)\|f^j_\delta\|_{L^q} \to 0.
\]
Combining this with (3.11) shows that the gradient Young measure generated by \( \{ v_j^3(\delta x) \} \) coincides with \( \nu \).

\[ \square \]

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