MULTIPHASE LAMINATES OF EXTREMAL EFFECTIVE CONDUCTIVITY IN TWO DIMENSIONS

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Abstract. This paper deals with two-dimensional composites made of several isotropic linearly conducting phases in prescribed volume fractions. The primary focus is on the three-phase case; the generalization to a larger number of phases is straightforward.

A class of high- but finite-rank laminates is introduced. The laminates saturate the known inequality bounds — due to the work of Hashin and Shtrikman, Lurie and Cherkaev, Tartar, and Murat and Tartar — on the effective conductivity tensor of any composite. These bounds depend only on the constituent material properties and volume fractions and not on the placement of these materials in the composite. The bounds are known not to be optimal for all admissible choices of the conductivities and volume fractions. However, they are now known to be realizable in a much larger range of these parameters than was previously known.

The range of effective properties of our multiphase laminates strictly includes those corresponding to the composites found earlier by Milton and Kohn, Lurie and Cherkaev, and Gibiansky and Sigmund. The new optimal laminates are found in a systematic fashion by satisfying sufficient conditions on the fields in each layer. This leads to a simple algorithm for generating optimal laminates.

In addition a new supplementary bound for multiphase structures is also proven which must be satisfied by composites with smooth interfaces.

http://dx.doi.org/10.1016/j.jmps.2006.12.003

1. Introduction

In this paper, we discuss new results for the long-standing problem of the optimality of the so-called “translation bounds”\textsuperscript{1} on the effective properties of two-dimensional composites of several isotropic conducting materials in prescribed proportions. The bounds are known not to be optimal for all choices of parameters — the constituent conductivities and prescribed proportions. However, it is still not clear what the conditions of optimality are.

\textsuperscript{1}The bounds are also frequently referred to as the Hashin-Shtrikman bounds, as they generalize the pioneering work of Hashin and Shtrikman (1962).
In this paper, we show that the bounds are optimal for a much larger range of parameters than was previously known.

The primary thrust of the paper is two-fold. First, we introduce new optimal structures which prove that the translation bounds are optimal for anisotropic composites outside the region of parameters previously known. Second, we introduce a general algorithm for constructing these structures and understanding their optimality by looking at sufficient conditions on the local electrical fields.

In Section 2, we recall the notions of periodic homogenization and effective tensors for the special case of two-dimensional, multi-material, conducting composites. We also pose the G-closure problem (sometimes called the $G_m$-closure problem in the literature), which characterizes the set of effective tensors available to us with several given materials in fixed proportions parameterized by a vector, $m$, of volume fractions.

In Section 3, we discuss known bounds on the G-closure, $G$, prior to the results of this paper. The set $G$ can be associated with a compact subset of $\mathbb{R}^2$ by mapping tensors in $G$ to their eigenvalue pairs. The bounds are divided into two categories: inner bounds and outer bounds. An outer bound is a subset of the plane $\mathcal{B} \supseteq G$. The outer bound we discuss in this paper is the intersection of the Wiener bound (also known as the harmonic and arithmetic mean bound) and the translation bound discussed in Section 3.1. An inner bound on the other hand, is a subset of the plane $\mathcal{L} \subseteq G$. Inner bounds are constructed by exhibiting a set of structures whose effective tensors form the set $\mathcal{L}$. In this paper, we do not attempt to improve the outer bound in is full generality (though we do prove a limit on the optimality of the bound for a special class of structures in Section 6). Instead, we produce a new inner bound which includes previously known inner bounds on the G-closure.

In Section 4, we construct this improved inner bound, using a class of finite- but high-rank laminate structures. In particular, we extend the known region of optimality of the translation bound. For definiteness, we primarily discuss the lower bound defined by (12) for three-material mixtures. The upper bound is dealt with analogously. The generalization to four and more materials is discussed in Section 5.5. Generalizations to other problems such as two-dimensional linear elasticity and three-dimensional linear conductivity are possible and will be discussed in other papers.

The result of the section is that the “translation bound” is now known to be optimal in the anisotropic case for a wider range of volume fractions $m_1, m_2, m_3$ ($m_1 + m_2 + m_3 = 1$), where $m_i > 0$ is the fixed relative amount of the $i$th material in the three-material G-closure problem. We assume the materials have isotropic conductivity tensors $K_i = k_i I$ for $i = 1, 2, 3$ where $I$ is the identity matrix. We label the materials so that $0 < k_1 < k_2 < k_3$. For the previous results to be applicable (see Section 3.2.2 and Theorem 1 in particular), the condition $m_1 \geq 2\Theta(1 - m_2)$ must be satisfied. The constant $\Theta$ is defined in (17) and depends only on the material properties $K_i$. In this
section, we introduce new anisotropic structures which are optimal for the translation bound (12) with an improved applicability condition:

\[
2\Theta(\sqrt{m_2 - m_2}) < m_1 < 1.
\]

(Note that \(\sqrt{m_2 - m_2} < 1 - m_2\) if \(m_2 \in (0, 1)\).)

As a first illustration of the results described in Section 4, consider Figures 1–2, which represent the outer bounds and their optimality in the plane of eigenvalues of an effective tensor \(K^*\). The figures represent the case

\[
k_1 = 1, \quad k_2 = 2, \quad k_3 = 5
\]

and

\[
m_1 = 0.4, \quad m_2 = 0.01, \quad m_3 = 0.59.
\]

We have chosen an example where \(m_2\) is small to illustrate the extremeness of the improvement in this case. While the amount of improvement depends on the values of the parameters, our results always improve on previously known results if the volume fractions satisfy (1).

**Figure 1.** (left) The bounds (10)-(13) with previously known optimal points. (right) a magnification of the upper-left corner.

First consider Figure 1. The thin dashed lines represent the Wiener bounds (10) and (11) while the thin curved lines represent the translation bounds (12) and (13). The thick portion of the upper bound represents known optimal points on the bound as proved in Milton and Kohn (1988) (see Theorem 1). The single isotropic point marked on the lower bound is the point proved optimal in Gibiansky and Sigmund (2000) (see Theorem 2). The thick portions of the Wiener bounds represent the optimal structures described in Cherkaev and Gibiansky (1996) (see Theorem 9). The right side of the figure shows a magnification of the top-left corner in order to show that the Wiener bounds are tighter than the translation bounds near this corner and that the Cherkaev-Gibiansky structures do not quite reach the translation bounds. Figure 2 shows the status after the results of the present paper are added.
In Section 5, we illustrate an algorithm for constructing the structures of the previous section by examining the local electrical fields in each constituent material. This approach is quite different from the traditional methods, which rely on physical intuition and/or numerical optimization to suggest structures for a given external loading and fixed volume fractions. Instead, the approach offered in this section leaves the external loading and volume fractions free and instead uses standard techniques for solving differential inclusions to produce structures consistent with the pointwise field requirements. In this sense, our approach is more systematic than traditional methods. After a composite has been produced in this fashion, a straightforward calculation finds the external loading and volume fraction constraints for which the structure is optimal. The algorithm resembles the methods used to solve a number of problems involving gradients supported on certain sets of matrices. We refer the reader to M"uller (1999); Dolzmann (2003); Conti et al. (2005a,b) for some examples.

Section 6 addresses the optimality of the translation bounds from a different direction. It is known that the bounds cannot be optimal for all values of the volume fractions. However, the exact conditions for optimality are still unknown. In this section, we consider the lower bound (12) for three materials. We derive an auxiliary inequality bound that a special type of structure (periodic structures with smooth interfaces) must satisfy if it attains the bound. The auxiliary bound limits the degree of anisotropy of such a structure. In particular, we show that no periodic structure with smooth interfaces can attain the bound if

$$m_1 < 2\Theta(\sqrt{m_2} - m_2)$$

(compare to (1)). While this does not answer the optimality question in general, we include this section for two reasons. First, the $T^2$-structures — which we describe in Section 4 — satisfy the bound as equality. Secondly, the bound is a useful application of the field analysis discussed in Section 5.
Indeed, the bound is obtained exactly by finding an inequality that the fields in the special structures must satisfy pointwise. The bound can also be applied to finite-rank laminates using properties of correctors as described in Briane (1994). However, no proof is currently known for general structures.

2. Problem and notations

In this section, we recall the problems of homogenization and the G-closure, fixing the notation. References for this section can be found, for example, in Dal Maso (1993); Braides and Defranceschi (1998); Cherkaev (2000); Allaire (2002); Milton (2002).

2.1. Multi-material conducting mixtures. Consider a two-dimensional, periodic, multi-material structure. The unit periodicity cell \( \Omega = [0, 1]^2 \) is partitioned into \( N \) disjoint sets \( \Omega_1, \ldots, \Omega_N \) such that

\[
\bigcup_{i=1}^{N} \Omega_i = \Omega.
\]

The relative volume fractions of each part, \( m_i = |\Omega_i| \) (where \( |\Omega_i| \) represents the area of the set \( \Omega_i \)), satisfy

\[ m_i \geq 0 \quad \forall i = 1, \ldots, N, \quad \sum_{i=1}^{N} m_i = |\Omega| = 1. \tag{2} \]

The \( \Omega_i \) are assumed to be filled by materials with isotropic conductivity tensors

\[ K_i = k_i I \quad \text{for } i = 1, \ldots, N \tag{3} \]

where \( I \) is the two-by-two identity matrix. We assume the conductivities are ordered so that

\[ 0 < k_1 < \cdots < k_N. \tag{4} \]

The conductivity equations applied to the periodicity cell are written as

\[ \text{div}(K(x)\nabla u(x)) = 0 \text{ in } \Omega, \quad \int_{\Omega} \nabla u(x) \, dx = e, \quad x = (x_1, x_2) \in \mathbb{R}^2 \tag{5} \]

where \( K : \Omega \to \{K_1, \ldots, K_N\} \) is the conductivity tensor defined by

\[
K(x) = \begin{cases} 
K_1 & \text{if } x \in \Omega_1, \\
\vdots \\
K_N & \text{if } x \in \Omega_N,
\end{cases} \tag{6}
\]

\( K_1, \ldots, K_N \) are given by (3), and where \( e \) is the prescribed average field induced by distant external sources.
Assume that the periodicity cell with material layout defined by $K(x)$ is subject to the average field $e$. The energy stored in the material is defined as

$$W(K, e) = \inf_{u \in H^1_\#(\Omega)+e-x} \int_{\Omega} \nabla u(x) \cdot K(x) \nabla u(x) \, dx$$

where $H^1_\#(\Omega)$ is the space of locally $H^1$ functions on $\mathbb{R}^2$ which are $\Omega$-periodic and have zero mean. The infimum is taken over functions which can be split into an affine part, $e \cdot x$, plus a periodic oscillating part:

$$u(x) = e \cdot x + \text{osc}(x), \quad \int_{\Omega} \nabla u(x) \, dx = e.$$

Notice that the affine part, $e \cdot x$, is prescribed by the loading. The minimization is taken over the variable, oscillating part, $\text{osc}(x)$.

The structure defined by the partition $\Omega_i$ is associated with its effective tensor $K_{\text{eff}}$, the conductivity tensor of homogeneous material that stores the same energy as the mixture under the same homogeneous loading. That is,

$$e \cdot K_{\text{eff}} e = \inf_{u \in H^1_\#(\Omega)+e-x} \int_{\Omega} \nabla u(x) \cdot K(x) \nabla u(x) \, dx \quad \forall e \in \mathbb{R}^2.$$

In order to completely determine $K_{\text{eff}}$, it suffices to consider the response of the same structure to two orthogonal loadings

$$e = r_1 e_1 = r_1 (1, 0)^T \quad \text{and} \quad e = r_2 e_2 = r_2 (0, 1)^T$$

where $r_1, r_2 \in \mathbb{R}$ are the magnitudes of the loadings and $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$. The response refers to the sum of the energies of the separate loadings:

$$W(K, r_1 e_1) + W(K, r_2 e_2) = \lambda_1 r_1^2 + \lambda_2 r_2^2, \quad \forall r_1, r_2 \in \mathbb{R}$$

where $\lambda_1$ and $\lambda_2$ are the eigenvalues of $K_{\text{eff}}$. This functional can be conveniently rewritten in terms of two-by-two matrices. We write $E = \text{diag}(r_1, r_2)$. Given any pair of potentials $U = (u_1, u_2)$, we define the gradient matrix as the matrix whose rows are the gradients of $u_1$ and $u_2$:

$$DU = (DU_{ij}), \quad DU_{ij} = \frac{\partial u_i}{\partial x_j}, \forall i, j = 1, 2.$$

The sum of energies (8) can then be written as

$$W(K, E) = \inf_{U \in H^1_\#(\Omega)^2+Ex} \int_{\Omega} \langle DU(x) K(x), DU(x) \rangle \, dx$$

where $\langle \cdot, \cdot \rangle$ is the inner product defined on two-by-two matrices by

$$\langle A, B \rangle = \text{tr}(AB^T).$$

The effective tensor $K_{\text{eff}}$ is the unique (symmetric) tensor satisfying the relation

$$\langle E K_{\text{eff}}, E \rangle = \inf_{U \in H^1_\#(\Omega)^2+Ex} \int_{\Omega} \langle DU(x) K(x), DU(x) \rangle \, dx \quad \forall E \in \mathbb{R}^{2 \times 2}.$$
The G-closure. We think of the volume fractions, \( m_i = |\Omega_i| \), and the material properties \( k_i \) as parameters of the problem. If \( m = (m_1, \ldots, m_N) \) and \( k = (k_1, \ldots, k_N) \) satisfy (2) and (4), then we say \( m \) and \( k \) are admissible parameters for the G-closure problem. The closure of the set of all possible \( K^{\text{eff}} \) available for parameters \( m \) and \( k \) is called the G-closure. Specifically, the G-closure \( G(m; k) \) is defined as

\[
G = G(m; k) = \{ K^{\text{eff}} : K \text{ as in (6); } |\Omega_i| = m_i, \ i = 1, \ldots, N \}.
\]

Observe that the variable in this problem is the partition of \( \Omega \) into the \( \Omega_i \), since \( K \) and thus \( K^{\text{eff}} \) are defined through the \( \Omega_i \). Each partition defines a material structure through (6). The question is: what is the set of possible \( K^{\text{eff}} \) which can be obtained by some partition into \( \Omega_i \) subject to the volume constraints \( |\Omega_i| = m_i \)? Note that for a given \( K^* \in G(m; k) \), there need not exist a partition of \( \Omega \) into \( \Omega_i \) and associated \( K \) as in (6) such that \( K^{\text{eff}} = K^* \). Indeed, one often shows that a particular \( K^* \) lies in the G-closure by finding a sequence of structures \( \{K^*_\} \) such that \( K^*_{\text{eff}} \to K^* \). We use the notation \( K^{\text{eff}} \) versus \( K^* \) to distinguish these concepts.


The set \( G(m; k) \) is known to be a closed and bounded subset of the two-by-two symmetric matrices, \( \mathbb{R}^{2\times2}_{\text{sym}} \). It is also rotationally invariant, so it suffices to consider the projection of the set into the two-dimensional plane of eigenvalues. As mentioned in the introduction, there is a two-fold strategy to characterize \( G(m; k) \): inner and outer bounds. By an outer bound, we mean a set \( B = B(m; k) \) such that \( B(m; k) \supseteq G(m; k) \). On the other hand, by an inner bound, we mean a set \( L = L(m; k) \) such that \( L(m; k) \subseteq G(m; k) \). Of course, if the sets \( L \) and \( B \) can be constructed so that \( L = B \), the we have characterized the entire G-closure. In this section, we discuss known inner and outer bounds on the G-closure. As we mentioned in the introduction, this paper does not improve the outer bounds in general, but instead produces a larger inner bound for admissible parameters \( m \) and \( k \) satisfying certain conditions.

3.1. Outer bounds. In this paper, we consider the outer bound \( B \) defined in the eigenvalue plane by a set of inequalities. In particular, any tensor \( K^* \in G(m; k) \) satisfies the following inequality bounds which depend only on the parameters \( m \) and \( k \). The set \( B(m; k) \) of two-by-two symmetric tensors which satisfy these inequalities forms an outer bound on the G-closure.

1. The Wiener bounds are
\[ \lambda_{\min}(K^*) \geq \left( \sum_{i=1}^{N} \frac{m_i}{k_i} \right)^{-1}, \]
\[ \lambda_{\max}(K^*) \leq \sum_{i=1}^{N} m_i k_i. \]

where \( \lambda_{\min}(K^*) \) and \( \lambda_{\max}(K^*) \) are the minimum and maximum eigenvalues of \( K^* \) respectively.

2. The translation bounds are

\[ \frac{\text{tr } K^* - 2k_1}{\det K^* - k_1^2} \leq 2 \sum_{i=1}^{N} \frac{m_i}{k_i + k_1}, \]
\[ \frac{\text{tr } K^* - 2k_N}{\det K^* - k_N^2} \geq 2 \sum_{i=1}^{N} \frac{m_i}{k_N + k_i}. \]

The translation bounds are not optimal for all values of the parameters \( m \) and \( k \). Intuitively, we see this from the fact that the formulas for the bounds still depend on \( k_1 \) (respectively \( k_N \)) when \( m_1 = 0 \) (respectively \( m_N = 0 \)). Indeed, for \( m_1 \) or \( m_N \) near 0, there are better bounds (see Talbot et al. (1995); Nesi (1995)). Despite several results in the area, it is still not known for which parameters the translation bounds are optimal. In the rest of this paper, we address this issue and extend the known range of parameters for which they are.

3.2. Inner bounds. In this section we summarize the known inner bounds on the G-closure prior to this paper. These bounds are found by proving that certain tensors \( K^* \) lie in \( G(m; k) \). One way to construct such a tensor is to exhibit a partition of \( \Omega \) into subsets \( \Omega_i \) with \( |\Omega_i| = m_i \) so that with \( K \) defined as in (6), one has \( K_{\text{eff}} = K^* \). However, such “exact structures” are often difficult to construct. A simpler method is to construct a sequence of partitions \( \{\Omega_i^\epsilon\}_{\epsilon > 0} \) so that \( |\Omega_i^\epsilon| = m_i \) and the corresponding effective tensors converge: \( K_{\text{eff}}^\epsilon \to K^* \) as \( \epsilon \to 0 \). We know from homogenization theory that \( K^* \) is then in \( G(m; k) \). This technique is simpler than that of finding an exact structure because one can use simple formulas for computing \( K^* \) without explicitly computing any of the \( K_{\text{eff}}^\epsilon \). Laminates are a common example of such structures and are the structures we exploit in this paper.

We focus on optimal structures — that is, structures (such as laminates) which produce an effective tensor \( K^* \in \partial G(m; k) \). We call the sequence of partitions an optimal structure and the associated \( K^* \) an optimal point on the G-closure. This notion is based on the fact that the G-closure can be completely characterized by its boundary. Indeed, \( G(m; k) \) is closed, bounded and simply connected (see for example Cherkaev (2000)). The trick to finding optimal structures is to use both inner and outer bounds. If
we have $\mathcal{L} \subseteq G(m;k) \subseteq \mathcal{B}$ and if there exists $K^* \in \mathcal{L} \cap \partial \mathcal{B}$ then $K^*$ is optimal. Notice that for $\mathcal{B}$ defined by (10)-(12), $K^* \in \partial \mathcal{B}$ if and only if at least one of the inequalities (10)-(12) is satisfied as equality. To differentiate these cases, we will say that $K^*$ is optimal for whichever inequality is satisfied as equality.

Furthermore, we will sometimes need to distinguish the case when there exists an effective tensor $K_{\text{eff}} = K^* \in \partial \mathcal{B}$ from the case when there exists a limit of effective tensors $K'_{\text{eff}} \to K^* \in \partial \mathcal{B}$. In the former case, we will say the bound $\mathcal{B}$ is attainable at the point $K^*$. In the latter case, we will say the bound is optimal at the point $K^*$. Clearly, $\mathcal{B}$ is optimal at any point for which it is attainable. However, there are examples of bounds which are optimal but not attainable. The Appendix discusses such an example. This distinction is essentially only important in Section 6.

3.2.1. Two-material optimal structures. The case of two-material optimal structures ($N = 2$) is completely known due to the work of Hashin and Shtrikman (1962); Lurie and Cherkaev (1984); Tartar (1979, 1985). The optimal isotropic structures were constructed by Hashin and Shtrikman (1962). They used the coated spheres construction (see Figure 3a): a circle filled with $K_2$ surrounded by a concentric annulus filled with $K_1$. When this construction is placed into an infinite plane with the conductivity $K^*$, $K_1 \leq K^* \leq K_2$ and a constant field is applied at infinity, the volume fraction of the circle in the structure can be chosen to keep the outside field homogeneous. Thus, the effective conductivity of the structure is $K^*$. The periodicity cell $\Omega$ can then be filled with infinitely many homeothetic coated circles (on infinitely many length scales). Hashin and Shtrikman showed that the effective conductivity of such structure is optimal for the isotropic version of (12) (which they discovered as well).

![Figure 3. Two-material structures optimal for the translation bound.](image)

The optimal anisotropic two-material structures were found by Lurie and Cherkaev (1984) together with the translation bound (12) for $N = 2$ (see Figure 3b. The structures are iterated laminates.

**Remark 1.** *The figures in this paper should be thought of as schematic representation of the actual structures. In reality, the laminate structures...*
must have well-separated scales for the results to apply. For example, in Figure 3b, one should imagine that the vertical strips of $K_1$ and $K_2$ are interleaved at a scale $\epsilon^2$, while the horizontal strips of this laminate with $K_1$ are interleaved at a scale $\epsilon$. The effective tensor $K^*$ is obtained by sending $\epsilon \to 0$.

The lamination formula. Recall the lamination formula of two materials, $K_A$ and $K_B$, with volume fraction of $K_A$ equal to $m$ and normal of lamination given by $n$. The effective tensor of the laminate is given by

$$K^* = L(K_A, K_B, n, m) = mK_A + (1 - m)K_B - \mathcal{N}$$

where

$$\mathcal{N} = m(1 - m)(K_B - K_A)n[n^T(mK_B + (1 - m)K_A)n]^{-1}n^T(K_B - K_A)$$

The optimal two-material structures use an iteration of this formula (see Tartar (1985)). First $K_1$ and $K_2$ are laminated in some proportion, then this new “auxiliary” material is laminated with $K_1$ in the direction orthogonal to the first lamination. The effective properties can be computed via

$$K^* = L(K_1, L(K_1, K_2, n_1, c_1), n_2, c_2)$$

where $n_1$ and $n_2$ are orthogonal and $c_1, c_2 \in [0, 1]$. The fractions $c_1$ and $c_2$ are related by $(1 - c_1)(1 - c_2) = m_2$. These structures and their dual version (interchanging $K_1$ with $K_2$ and $m_1$ with $m_2$) describe all optimal structures (they are optimal for either (12) or (13)). Both classes of structures degenerate into laminates (for example, when $c_1 = 0, c_2 = 1 - m_2$).

**Remark 2.** While the optimal structures described above are sequences whose effective conductivities converge to a point on the bound, there do exist exact geometries whose effective conductivities attain each point of the two-dimensional translation bound. We refer the reader to Vigdergauz (1989, 1999); Grabovsky and Kohn (1995); Astala and Nesi (2003) for more details.

3.2.2. Multi-material optimal structures.

Milton’s structure. The first type of isotropic, multi-material ($N \geq 3$) structures which were proved optimal for the translation bound were described by Milton (1981) (see Figure 4a). His construction is as follows. The amount $m_1$ is split into two parts $m'_1$ and $m''_1$ so that the coated circles structures from $K_1$ and $K_2$ (in the proportions $\frac{m'_1}{m'_1 + m_2}$ and $\frac{m_2}{m'_1 + m_2}$, respectively) and from $K_1$ and $K_3$ (in the proportions $\frac{m''_1}{m''_1 + m_3}$ and $\frac{m_3}{m''_1 + m_3}$, respectively) have the same effective conductivity. Obviously, any mixture of these structures has the conductivity of each of them. This mixture is optimal for the bound (12) ($K^*$ is isotropic). All mixtures of this form clearly satisfy

$$K_1 \leq K^* \leq K_2$$
since the mixture of $K_1$ and $K_2$ must lie in this range. Such a construction is possible as long as there is enough of material $k_1$. Specifically, this construction requires that

$$m_1 \geq 2\Theta(1 - m_2)$$

where $\Theta$ is a constant defined below in (17). Similar structures are optimal for the opposite bound (13) with $K_3$ taking the role of the “coating”, and $K_1$ and $K_2$ the inclusions.

Lurie-Cherkaev multi-coated spheres. An inner bound for the $G$-closure problem was found in Lurie and Cherkaev (1985) by posing an additional assumption that the structure is of the type of multi-coated circles (see Figure 4b) and then solving the corresponding optimal control problem. Their construction is geometrically different from Milton’s, but the effective conductivities of both structures coincide in the range of parameters where (16) holds. The structures are not optimal for the translation bound if (16) is violated.

Optimal structures for the Wiener bounds. Cherkaev and Gibiansky (1996) introduced a class of three-material anisotropic structures. They have the property that they are optimal for one of the Wiener bounds (10) or (11), but not the other. For large enough $m_1$, the structures resemble those illustrated in Figure 7b, but with different volume fractions from those chosen in the following section. For smaller $m_1$, the outer layer of $K_1$ is replaced by a layer of $K_3$. In the Appendix, we use this construction to illustrate the difference between the attainability and optimality of an outer bound.

Milton-Kohn Matrix laminates. The matrix laminates introduced by Milton and Kohn (1988) combine the idea of Milton (1981) with the two-material anisotropic structures (15) (see Figure 4c). The amount $m_1$ of $K_1$ is divided into two parts, which are used to form two different mixtures of materials, one of $K_1$ and $K_2$ and the other of $K_1$ and $K_3$. The mixtures are chosen to be optimal for the corresponding two-material G-closure problem and have

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2The analogous two-material structures are geometrically impossible — that is, if one of the bounds is satisfied as equality for some $K^*$ then so is the other.
effective tensors given by

\[ K^* = L(K_1, L(K_1, K_2, n_1, c_1), n_2, c_2) \]
\[ K^{**} = L(K_1, L(K_1, K_3, n_3, c_3), n_4, c_4) \]

where

\[ n_1^T n_2 = 0, \quad n_3^T n_4 = 0, \quad c_1, c_2, c_3, c_4 \in [0, 1]. \]

From the results of two-material structures, we know both \( K^* \) and \( K^{**} \) are optimal structures. Furthermore, if the parameters can be chosen so that \( K^* = K^{**} \), then the linearity of the bounds (12) with respect to the volume fractions allows us to mix the two constructions together in any way we wish to obtain an optimal three-material structures. These constructions also require that (16) hold. These structures are more general than those previously discussed: they include anisotropic structures as well as isotropic. The results can be summarized as follows.

**Theorem 1** (Milton-Kohn). Let \( m = (m_1, m_2, m_3) \) and \( k = (k_1, k_2, k_3) \) be admissible parameters for the G-closure problem such that

\[ m_1 \geq 2\Theta(1 - m_2), \]

where

\[ \Theta = \frac{k_1(k_3 - k_2)}{(k_2 + k_1)(k_3 - k_1)} \leq \frac{1}{2}. \]

Then there exists a family of structures with the given volume fractions which are optimal for (12). The effective tensors of this family cover a connected subset of the translation bound curve which includes the isotropic point. The most anisotropic structure of this family has an effective tensor with eigenvalues given by

\[ \lambda_1 = \nu k_1 + (1 - \nu)k_2, \quad \lambda_2 = \frac{k_1k_2}{(1 - \nu)k_1 + \nu k_2} \]

where \( \nu \) is defined to be

\[ \nu = m_1 - \frac{2\Theta}{1 - 2\Theta} m_3. \]

**Remark 3.** Consult the thick, solid line in Figure 1 for an illustration of the family structures for the other bound, (13). The family in the theorem covers a segment of the bound near the isotropic point.

**Remark 4.** As in the two-material case, there also exist exact geometries which attain certain points of the translation bound. In this case, however, it is known that there exist G-closure parameters for which the bounds are not even optimal. Once again, we refer the reader to Astala and Nesi (2003) for more details on these exact structures.

After the paper by Milton and Kohn, the development slowed down. The combination of two facts: the known non-optimality of the translation bound
as \( m_1 \to 0 \) and the "natural" limit, \( K^* \leq k_2 I \), of the known optimal structures provocatively suggested that there were no other structures optimal for the bound. However, in a surprising development twelve years later, Gibiansky and Sigmund (2000) discovered new isotropic structures that are optimal for the translation bound for smaller values of \( m_1 \) than given in Theorem 1.

Gibiansky-Sigmund isotropic structures. Gibiansky and Sigmund (2000) announced a new construction that significantly increased the set of optimal points of the translation bounds, (12) and (13). Their structures were the surprising result of a numerical simulation. Using a "topology optimization" algorithm developed earlier by Sigmund, the authors searched for optimal structures by computer. We refer the reader to Figures 4–9 in their paper for examples of the fascinating structures selected by the procedure. In particular, Figure 6 in their paper illustrates a structure which violates the condition (16) of Theorem 1 but which numerically appears to satisfy the translation bound. When the authors attempted to replace the computer output with a similar, but simpler structure for which the effective properties could be analytically computed, the simplified structure was optimal for the translation bounds. In Figure 5, we illustrate a special case of their structure.

Instead of iterated laminates or coated spheres, they used a construction (also used earlier by Sigmund (2000)) which resembles the work of Marino and Spagnolo (1969). In the latter paper, the authors consider a completely different issue. However, they introduce (among other things), conductivities \( b(x) \) in \( d \) dimensions which take the form

\[
b(x) = b_1(x_1)b_2(x_2)\cdots b_d(x_d)
\]

and study some special cases. Roughly speaking, Gibiansky and Sigmund consider conductivities which have this form but on several different scales and only in an approximate sense.

Reinterpreting their results slightly, we divide the cell of periodicity into four rectangular subdomains. Two opposite squares are occupied by \( K_2 \) and \( K_3 \), and the remaining rectangles are filled with laminates from \( K_1 \) and \( K_3 \).

\[\text{Figure 5. Extremal structures of Gibiansky and Sigmund.}\]
The effective conductivity of the laminate depends on the volume fraction of materials in it. This conductivity (or, equivalently, the volume fractions in the laminate) is chosen in such a way that the conductivity equation (5) permits a separation of the variables if the average fields are homogeneous. Because of this feature, the solution is analytic, and so are the effective properties. Using Maple, the authors then found that the structures are optimal for the translation bound (12). The result is amazing because the structure is a mathematical approximation of a numerical approximation of the optimization problem. The authors also described more complicated structures that were optimal for larger values of \( m_1 \) and which coincided with the previously known structures at the point \( K^* = K_2 \). Their results are summarized by the following theorem, which lowers the minimum value of \( m_1 \) for which the bound (12) is optimal.

**Theorem 2** (Gibiansky-Sigmund). Let \( m = (m_1, m_2, m_3) \) and \( k = (k_1, k_2, k_3) \) be admissible parameters for the G-closure problem such that

\[
2\Theta(\sqrt{m_2} - m_2) \leq m_1 < 2\Theta(1 - m_2),
\]

where \( \Theta \) is given in (17). Then there exists an isotropic structure with the given volume fractions and optimal for the bound (12).

**Remark 5.** The results of Gibiansky and Sigmund raise an interesting question. If the volume fractions satisfy the inequalities

\[
2\Theta(\sqrt{m_2} - m_2) \leq m_1 < 2\Theta(1 - m_2),
\]

is the isotropic point on the translation bound attainable? In particular, is it possible to find an explicit formula for the partition suggested by the computer-generated structure which delivers an effective tensor on the translation bound?

In the next section, we introduce a class of anisotropic laminate structures which contains structures with the same properties as the those of Gibiansky and Sigmund. In Section 5, we analyze the fields in optimal structures and obtain a clear picture of their features.

### 4. New optimal structures

In this section, we construct a family of optimal laminate structures. These structures are all particular cases of the structure illustrated in Figure 6, an orthogonal laminate of high rank with six design parameters and five well-separated scales (see Remark 1). In this section, we choose the structural parameters so that the structure satisfies the translation bound (12) as equality. We begin with degenerate cases and work toward the structure in full generality. We delay until Section 5 the discussion of why we should expect such a structure to be optimal. The reader more interested in the reasons the structures are optimal than in the structures themselves is invited to skip directly to Section 5.
A convenient change of variable. The material $K_1$ and its volume fraction $m_1$ play a special role in the bound (12) and in the associated optimal structures. For this reason, it is convenient to introduce (and fix) the relative fractions of the other two materials. Given $m_1, m_2, m_3 > 0$ with $m_1 + m_2 + m_3 = 1$, define $p \in (0, 1)$ by

$$p = \frac{m_2}{m_2 + m_3}. \tag{18}$$

Note that it follows that

$$1 - p = \frac{m_3}{m_2 + m_3} \quad \text{and} \quad \frac{m_2}{m_3} = \frac{p}{1-p}.$$

Using $p$-notation, the translation bound (12) for three material mixtures is rewritten as

$$\frac{1}{2} \cdot \frac{\text{tr} K^* - 2k_1}{\det K^* - k_1^2} \leq \frac{m_1}{2k_1} + (1 - m_1) \left( \frac{p}{k_2 + k_1} + \frac{1-p}{k_3 + k_1} \right).$$

We think of $p \in (0, 1)$ as a parameter of the problem. With $p$ fixed, we write the requirement that a structure is optimal for (12) as

$$m_1 = \frac{1}{2} \cdot \frac{\text{tr} K^* - 2k_1}{\det K^* - k_1^2} - \left( \frac{p}{k_2 + k_1} + \frac{1-p}{k_3 + k_1} \right). \tag{19}$$

T-structures. The simplest optimal three-material structure in this section is the $T$-structure. It is assembled as a sequence of laminates which depends upon two parameters. First, $K_1$ and $K_3$ are laminated with normal in the $x_1$-direction. Then, the resulting structure is laminated with $K_2$ with the normal in the $x_2$-direction. Figure 7a illustrates the construction of the $T$-structure. The effective properties are found by iterating the formula (14)

$$K_T = L \left( K_2, L \left( K_1, K_3, n_1, \frac{m_1}{m_1 + m_3} \right), n_2, m_2 \right)$$

where $n_1 = (1, 0)^T$ and $n_2 = (0, 1)^T$. 
Theorem 3. Let \( t \in (0, 1) \) and \( 0 < k_1 < k_2 < k_3 \). Then there exist volume fractions \( m_1, m_2, m_3 > 0 \) so that \( p = t \) with \( p \) given by (18) and such that the T-structure with these volume fractions is optimal for the translation bound (12). The values of the volume fractions are

\[
m_1 = \frac{\Theta(1-p)}{1-p\Theta}, \quad m_2 = \frac{p(1-\Theta)}{1-p\Theta}, \quad m_3 = \frac{(1-p)(1-\Theta)}{1-p\Theta}
\]

where \( \Theta \) is defined in (17). The eigenvalues \( \lambda_1, \lambda_2 \) of the optimal T-structure are computed to be

\[
\lambda_1 = \frac{(1-\Theta)p k_2 + (1-p)k_3 \beta}{(1-\Theta)p + (1-p)}
\]

\[
\lambda_2 = \frac{(1-\Theta)p k_2 + (1-p)k_2}{(1-\Theta)p + (1-p)\beta}
\]

where

\[
\beta = \frac{k_2 + k_1}{k_3 + k_1}
\]

It may seem surprising that we have found that there is always an optimal T-structure for any \( p \). This happens because we consider structures with fixed relative volume fractions of \( K_2 \) and \( K_3 \) but with arbitrary fraction of \( K_1 \).

Coating preserves optimality. In order to describe the variety of the optimal structures, we make the following observation.

Theorem 4 (The Coating Principle). If a structure with effective conductivity \( K^* \) is optimal for the translation bound (12), then all structures obtained by laminating it with material \( K_1 \) are also optimal for (12), though with different volume fractions. The laminating can be iterated several times with various normals so that the original structure is “coated” by \( K_1 \).

Proof. It is enough to apply the lamination formula (14) to \( K^* \) (with volume fractions \( m_1, m_2 \) and \( m_3 \)) and \( K_1 \), specifying the normal of lamination, \( n \),
and the volume fraction, \( c \), of \( K_1 \). This lamination produces a new material with effective tensor
\[
K^{*t} = L(K^*, K_1, n, c).
\]
Substituting \( K^{*t} \) into the bound (12) along with the updated volume fractions
\[
m'_1 = 1 - c + cm_1, \quad m'_2 = cm_2, \quad m'_3 = cm_3
\]
and using the fact that \( K^* \) satisfies the bound as equality, one can verify that \( K^{*t} \) also satisfies the bound as equality, with the new volume fractions. □

The theorem states that laminating optimal structures for the translation bound with \( K_1 \) preserves the optimality, though it changes the volume fractions. This observation allows us to restrict ourselves to the description of only extremal structures that attain the bound (12). By extremal structures, we mean structures that contain the minimal amount of \( K_1 \).

Example: the optimality of two-material matrix laminates. As a particular example, the coating principle can be used to prove the optimality of the two-material second-rank laminates discussed in Section 3.2.1. This case can be considered a special case of the three-material problem with
\[
p = \frac{m_2}{m_2 + m_3} = 1
\]
(that is, \( m_3 = 0 \)). Begin with \( \Omega \) filled with pure \( K_2 \) so that \( m_1 = 0 \) and \( m_2 = 1 \). These volume fractions clearly satisfy the requirement on the ratio \( p \). Furthermore, it is easy to check that (19) holds for this structure since \( K^* = K_2 \) so
\[
\frac{1}{2} \frac{\text{tr} K^* - 2k_1}{\det K^* - k_2^2} - \left( \frac{p}{k_2 + k_1} + \frac{1-p}{k_3 + k_1} \right) = \frac{1}{2} \frac{2k_2 - 2k_1}{k_2^2 - k_1^2} - \left( \frac{1}{k_2 + k_1} + \frac{0}{k_3 + k_1} \right)
\]
\[
= \frac{1}{k_2 + k_1} - \frac{1}{k_3 + k_1} = 0 = m_1.
\]
Therefore, the block of pure \( K_2 \) is optimal for (12). Applying the coating principle once, we then find that any lamination of \( K_2 \) and \( K_1 \) is also optimal. Finally, applying the principle a second time, we find that the structures illustrated in Figure 3b are optimal.

The coating principle also plays an important role in the analysis of multimaterial mixtures. Notice that the coating changes the volume fractions, \( m_i \), but it preserves the value of \( p \). Since coating increases the value of \( m_1 \), the principle allows to look for the optimal structures with the lowest value of \( m_1 \). Every optimal structure generates a set of optimal coated structures. The set \( \mathcal{L}(K^*) \) of optimal structures obtained by coating is a domain in the plane of eigenvalues of the effective tensor \( K^* \). The two boundary components of this set correspond to the laminates of the anisotropic, generating material \( K^* \) and \( K_1 \) with normal parallel to one of the eigenvectors of \( K^* \).
To derive the equations for the boundaries, let $K^*$ be given with volume fractions described by $m_1$ and $p$. Let $\lambda_1$ and $\lambda_2$ be the eigenvalues of $K^*$.

The boundary of $L(K^*)$ is found from the lamination formula, (14), by laminating $K^*$ with $K_1$ with volume fractions $c$ and $1-c$, respectively:

$$K^{*1} = L(K^*, K_1, n, c)$$

where $n$ is parallel to an eigenvector of $K^*$. The laminate contains $K_1$ in the fraction $m_1' = 1 - c + cm_1$, and the value of $p$ is preserved, $p' = p$. If $n$ is chosen parallel to the eigenvector associated with $\lambda_1$, then the eigenvalues $\lambda_1'$ and $\lambda_2'$ of $K^{*1}$ are parameterized by $c$ as

$$B_1(K^*) = \left\{ \left( \frac{c}{\lambda_1} + \frac{1-c}{k_1} \right)^{-1}, c\lambda_2 + (1-c)k_1 \right\} : c \in [0, 1]$$

while if $n$ is parallel to the other eigenvector, the new eigenvalues are parameterized by $c$ as

$$B_2(K^*) = \left\{ c\lambda_1 + (1-c)k_1, \left( \frac{c}{\lambda_2} + \frac{1-c}{k_1} \right)^{-1} \right\} : c \in [0, 1]$$

We define $\Lambda(K^*) \subset \mathbb{R}^2$ to be the closed set bounded by $B_1(K^*) \cup B_2(K^*)$ defined in (21) and (22) where $\lambda_1$ and $\lambda_2$ are the eigenvalues of $K^*$. We define

$$L(K^*) = \{ K^{*1} \in \mathbb{R}_{sym}^{2x2} : K^{*1} \text{ has eigenvalues in } \Lambda(K^*) \}$$

where $\mathbb{R}_{sym}^{2x2}$ denotes the two-by-two symmetric matrices. The following is an immediate corollary of Theorem 4.

**Corollary 1.** Let $t \in (0, 1)$. Let $k = (k_1, k_2, k_3)$ and $m = (m_1, m_2, m_3)$ be admissible parameters of the G-closure problem such that $m_2 = t(m_2 + m_3)$. Suppose $K^* \in G(m; k)$. Then, for any effective tensor $K^{*1} \in L(K^*)$, there exist admissible volume fractions $n' = (n_1', n_2', n_3')$ such that $n_2' = t(n_2' + n_3')$ and such that $K^{*1} \in G(m'; k)$.

Coated T-structures. From the optimal T-structure, we obtain a set of optimal structures by coating with $K_1$. This set, $L(K_T)$, is shaded in the eigenvalue plane in Figure 8 for the parameters

$$k_1 = 1, \quad k_2 = 2, \quad k_3 = 5, \quad p = \frac{1}{60}.$$ 

It is convenient to represent an anisotropic material by two symmetric points $(\lambda_1, \lambda_2)$ and $(\lambda_2, \lambda_1)$ in the plane of eigenvalues to avoid ordering them. Particularly, the optimal T-structure is represented by two points, both labeled $K_T$. The domain $L(K_T)$ of optimal structures as defined in (23) is the union of two lens-shaped regions in the plane. The boundaries of this set are the laminate curves. Recall that rather than fixing volume fractions, we fix the value $p$ which in turn fixes the ratio of $m_2$ to $m_3$. The figure also includes some dotted curves of constant volume fraction. Those closer to $K_1$ indicate larger values of $m_1$ than those farther away. Any point where...
one of these curves intersects the region $\mathcal{L}(K_T)$ is an optimal point for the translation bound (12) with the volume fractions given through $m_1$ and $p$.

**Remark 6.** Observe the change in topology of the intersection of constant volume fraction curves with $\mathcal{L}(K_T)$. For large values of $m_1$, the intersection is a connected portion of the curve. As $m_1$ decreases, the intersection suddenly becomes disconnected (specifically, when the constant volume fraction curve passes through the point $K_2$). Letting $m_1$ continue to decrease, one sees that the two connected components of the intersection shrink to points and then vanish (when the curve passes through the points $K_T$). For $m_1$ lower than this, the intersection is empty.

The two outer curves from $K_T$ to $K_1$ represent the most anisotropic structures of the class of coated T-structures. The pair of eigenvalues $\lambda_1, \lambda_2$ along these curves are parameterized by

\begin{align}
\lambda_1 &= \left( \nu \frac{1}{k_1} + (1 - \nu) \right) \left( \frac{(1 - \Theta)p + (1 - p)}{(1 - \Theta)p k_2 + (1 - p) k_3 \beta} \right)^{-1} \\
\lambda_2 &= \nu k_1 + (1 - \nu) \left( \frac{(1 - \Theta)p k_2 + (1 - p) k_2}{(1 - \Theta)p + (1 - p) \beta} \right)
\end{align}

where the constants $\Theta$ and $\beta$ are defined in (17) and (20). We can parameterize the inner curves analogously. The parameter $\nu \in [0, 1]$ along the outer
curve controls the amount of material \( K_1 \) added to the T-structure. The volume fractions of the final structure depending on \( \nu \) and \( p \) are given by

\[
m_1 = \nu + (1-\nu) \frac{\Theta(1-p)}{1 - p\Theta}, \quad m_2 = (1-\nu) \frac{p(1-\Theta)}{1 - p\Theta}, \quad m_3 = (1-\nu) \frac{(1-p)(1-\Theta)}{1 - p\Theta}.
\]

The more isotropic curves cross the line of isotropy at exactly \( \lambda_1 = \lambda_2 = k_2 \) and do so when

\[
\nu = \frac{(1-p)(k_3\beta - k_2)}{(1-\Theta)p(k_2 - k_1) + (1-p)(k_3\beta - k_1)}.
\]

At this point, we find that \( m_1 = 2\Theta(1 - m_2) \) (see Theorem 1!) The region of intersection (shaded darker) of the two lenses was proved to be optimal by Milton and Kohn (1988) using the laminated structures described in Section 3.2.2. The remaining portion of the region of \( L(K_T) \) represents the first new optimal structures of this paper. We will improve this region later. Observe that the only optimal isotropic structures found in \( L(K_T) \) were already known to be optimal. However, this construction has introduced a relatively large set of new optimal anisotropic structures. Furthermore, this construction proves the optimality of the bound (12) in a region of anisotropic points for a smaller value of \( m_1 \) than was previously known possible: \( m_1 \geq \Theta(1 - m_2) \) rather than \( m_1 \geq 2\Theta(1 - m_2) \). The coated T-structures are a generalization of the Milton-Kohn structures in the sense that the latter always have effective tensors in \( L(K_T) \). For reference, we have also indicated by the dashed line from \( K_2 \) to \( K_{GS} \) the optimal isotropic structures introduced by Gibiansky and Sigmund. This line intersects the set \( L(K_T) \) only at \( K_2 \).

\( T^2 \)-structures. We now enlarge the class of optimal structures with a set of structures which connects the points \( K_T \) and \( K_{GS} \) in Figure 8. We laminate the T-structure with a laminate from \( K_1 \) and \( K_3 \) in the orthogonal direction as illustrated in Figure 7c. The effective tensors of such \( T^2 \)-structures are found from the iterative procedure

\[
\begin{align*}
K_{T2} &= L(K_T, K_{13}', n_1, \omega_2), \\
K_T &= L(K_2, K_{13}, n_2, \omega_1), \\
K_{13}' &= L(K_1, K_3, n_2, \nu'), \\
K_{13} &= L(K_1, K_3, n_1, \nu).
\end{align*}
\]

Here, \( \nu \) and \( \omega_1 \) are the parameters of the original T-structure. \( \nu' \) is the relative fraction of \( K_1 \) in the additional \( K_1-K_3 \) laminate, and \( \omega_2 \) is the relative amount of the T-structure compared to the additional laminate in the final \( T^2 \)-structure.

The properties depend on four structural parameters: \( \nu, \nu', \omega_1, \omega_2 \) that all vary in \([0, 1]\) and are subject to the constraint that fixes \( p \). (Recall that \( m_1 \) is treated as a variable.) Observe that the \( T^2 \)-structures are a generalization of
the T-structures. (If $\omega_2 = 0$ in the equations above, then $K_{T_2} = K_T$.) Moreover, we show that they describe a curve of anisotropic structures between the T-structures and the isotropic structures of Gibiansky and Sigmund.

**Theorem 5.** Let $t \in (0, 1)$ and let admissible parameters $m = (m_1, m_2, m_3)$ and $k = (k_1, k_2, k_3)$ be given such that

$$p = \frac{m_2}{m_2 + m_3} = t$$

and

$$2\Theta \frac{-p + 2p^2\Theta - p\Theta + \sqrt{p^2\Theta^2 - 2p^2\Theta + p}}{(1 - 2p\Theta)^2} \leq m_1 \leq \frac{\Theta(1 - p)}{1 - p\Theta}.$$  

Then there exists a $T^2$-structure with the given volume fractions and optimal for the bound (12). These structures vary between the anisotropic $T$-structure and the isotropic point of Gibiansky and Sigmund (2000). The optimal volume fractions (see (26)) in the structure satisfy

$$\nu = \Theta, \quad \nu' = \omega_1 \Theta$$

where $\Theta$ is defined in (17) and $\omega_1, \omega_2$ satisfy

$$\omega_1 + \omega_2 = \frac{1}{\Theta}(m_1 + 2\Theta m_2), \quad \omega_1\omega_2 = m_2.$$  

The effective tensors have eigenvalues $\lambda_1, \lambda_2$, written in terms of $\omega_1$ and $\omega_2$ as

$$\lambda_1 = \frac{\omega_1 k_2(k_3 + k_1) + (1 - \omega_1)k_3(k_2 + k_1)}{\omega_2(k_3 + k_1) + (1 - \omega_2)(k_2 + k_1)},$$

$$\lambda_2 = \frac{\omega_2 k_2(k_3 + k_1) + (1 - \omega_2)k_3(k_2 + k_1)}{\omega_1(k_3 + k_1) + (1 - \omega_1)(k_2 + k_1)}.$$  

Notice that the relation between the effective properties of optimal mixtures is symmetric to the interchanging of $\omega_1$ with $\omega_2$ in spite of the nonsymmetric iterative procedure.

When $m_1$ equals its upper bound in (27), then one of $\omega_1$ or $\omega_2$ must be equal to 1. Thus, the $T^2$-structure degenerates into the T-structure for this volume fraction. On the other hand, when $m_1$ equals its lower bound, then we find

$$\omega_1 = \omega_2 = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{m_1}{2\Theta}}$$

and we obtain an isotropic structure whose volume fractions satisfy the equality $m_1 = 2\Theta(\sqrt{m_2 - m_2})$ (compare to Theorem 2!). The set of $T^2$-structures is a generalization of both the T-structures and the Gibiansky-Sigmund structure with minimal amount of $m_1$ (see Figure 9).

**Remark 7.** We show in Section 5 how to find these structures by analysis of the fields. In particular, we show how the parameters $\omega_1, \omega_2, \nu$ and $\nu'$ are naturally determined and explain the remarkable properties of the local fields in optimal structures that lead us to the construction.
The set of optimal structures. Applying the coating principle to the extremal $T_2$-structures, we obtain a variety of optimal structures because each $T_2$-structure can be coated, increasing the amount $m_1$ but keeping relative fraction $p$. The union of all the $L(K_{T_2})$ (see (23)), for each $K_{T_2}$ with the given value of $p$, forms a set of structures optimal for the translation bound (12). This set is illustrated in the eigenvalue plane in Figure 9 for parameters

$$k_1 = 1, \quad k_2 = 2, \quad k_3 = 5, \quad p = \frac{m_2}{m_2 + m_3} = \frac{1}{60}.$$ 

The set of optimal structures is bounded by the solid boundary, which is the union of coated $T$-structures (the curves between $K_1$ and $K_T$) and the $T_2$-structures (the curve passing through $K_{GS}$). The closed region bounded by the dashed lines represent the previously known optimal structures of Milton and Kohn, and Gibiansky and Sigmund.

Curves of constant volume fraction are indicated by the dotted lines. In particular the curve passing through $K_T$ represents the case $m_1 = \Theta(1 - m_2)$ while the curve passing through $K_{GS}$ represents the case $m_1 = 2\Theta(\sqrt{m_2} - m_2)$. Recalling Remark 6, we see that the topology of the intersection of the curves $m_1 = \text{const}$ and the new region of optimal structures remains connected for all volume fractions satisfying $m_1 > 2\Theta(\sqrt{m_2} - m_2)$. As $m_1$ decreases below this amount, the intersection shrinks to a single point and then vanishes for all smaller $m_1$.

The problem with fixed volume fractions. Until now, we have considered a problem of fixed $p = m_2/(m_2 + m_3)$. The classical G-closure problem, however, asks that we fix the volume fractions. Obtaining this information from our results is quite straightforward because the pair $(m_1, p)$ uniquely determines all volume fractions.

As an example, we find optimal structures in $G(m;k)$ for the parameters

$$k_1 = 1, \quad k_2 = 2, \quad k_3 = 5$$

and

$$m_1 = 0.4, \quad m_2 = 0.01, \quad m_3 = 0.59$$

as in Figures 1–2. To accomplish this, we need only examine Figure 9 (notice that $m_2/(m_2 + m_3) = 1/60$) and the corresponding figure for the upper bound with $m_2/(m_1 + m_2) = 1/41$, Figure 10. The optimal points of the lower bound marked by the thick portion of the curve in Figure 2 are the intersection of the curve of constant $m_1 = 0.4$ with the optimal region shown in Figure 9. The dot in Figure 1 marks the point where this curve intersects the dashed Gibiansky-Sigmund line. Similarly, the optimal points marked by the thick curve on the upper bound in Figure 2 are where the line of constant volume fraction $m_3 = 0.59$ intersects the set of optimal points in Figure 10. The thick portion of the upper bound in Figure 1 marks the intersection with the Milton-Kohn region.

Look again at Figure 9. As long as $m_1 \geq 2\Theta(\sqrt{m_2} - m_2)$ (For the given parameters, $m_1 \geq 0.05488.$), the intersection of the curve $m_1 = \text{const}$ and
Figure 9. Optimal points for the lower translation bound for $\frac{m_2}{m_2 + m_3} = \frac{1}{60}$. The curve between $K_T$ and $K_{GS}$ corresponds to the optimal $T^2$-structures. The curve between $K_T$ and $K_1$ corresponds to the coated $T$-structures illustrated in Figure 7b. The dashed line from $K_2$ to $K_{GS}$ corresponds to the structures of Gibiansky and Sigmund. The dashed curve passing through $K_1$ and $K_2$ forms the boundary for the structures of Milton and Kohn.

The region of attainable points is a connected subset of the curve which includes the isotropic point. Thus, the intersection is uniquely defined by the most anisotropic point of this subset. For $m_1 \geq \Theta(1 - m_2)$ (For the given parameters, $m_1 \geq 0.2469$) this most anisotropic point is a coated $T$-structure. For $m_1 \leq \Theta(1 - m_2)$, it is a $T^2$-structure. We summarize this in the following theorem.

**Theorem 6.** Let the volume fractions $m_1, m_2, m_3 > 0$, and material properties $0 < k_1 < k_2 < k_3$ be given admissible parameters. Define $p$ as in (18) and $\Theta$ as in (17). Then the following hold.

(i) If $m_1 > \Theta(1 - m_2)$, then (12) is optimal. There exists a set of optimal points on the bound which includes the isotropic point and whose most anisotropic member is that given by the coated $T$-structure with eigenvalues (24) and (25) where $\nu$ is chosen to satisfy the volume fraction constraints:

$$\nu = \frac{1 - p\Theta}{1 - \Theta} \left( m_1 - \frac{\Theta(1-p)}{1 - p\Theta} \right).$$

(ii) If $2\Theta(\sqrt{m_2} - m_2) \leq m_1 \leq \Theta(1 - m_2)$, then (12) is optimal. There exists a set of optimal points on the bound which includes the
Applicability: volume fractions. In Figure 11, we illustrate the difference between the applicability of Theorem 1 and Theorem 6 in the case of isotropic structures. In both figures we take \( k_1 = 1, k_2 = 2, k_3 = 5 \). By the definition of admissible volume fractions, we know that in the \( m_1m_3 \)-plane (note that \( m_2 = 1 - m_1 - m_3 \)), the volume fractions are constrained to the region bounded by the axes \( m_1 = 0 \) and \( m_3 = 0 \) and by the line \( m_1 + m_3 = 1 \).

Consider Theorem 1 for the lower bound (12) which we illustrate on the left of Figure 11. It implies that there is an isotropic structure which attains the bound if \( m_1 \geq 2\Theta(1 - m_2) \) where \( \Theta \) is defined in (17). We have indicated the line \( m_1 = 2\Theta(1 - m_2) \) by the thick solid curve passing from the origin \((m_2 = 1, m_1 = m_3 = 0)\) to the line \( m_1 + m_3 = 1 \). (The point of intersection is \( m_2 = 0, m_1 = 2\Theta, m_3 = 1 - 2\Theta \).) For any admissible volume fractions which lie to the right of this curve, Theorem 1 implies that there exists an isotropic structure which is optimal for the lower bound (12). The thick dashed line gives similar information for the upper bound (13). For any admissible volume fractions lying above this curve, there exists an isotropic structure which satisfies the upper bound as equality. In particular, the
Figure 11. Domain of applicability of Theorem 1 (left) and Theorem 6 (right) in terms of volume fractions. (Here we consider the isotropic case for $k_1 = 1, k_2 = 2, k_3 = 5$.) The volume fractions are physically constrained to the region bounded by the axes $m_1 = 0$ and $m_3 = 0$ and the line $m_1 + m_3 = 1$. The thick solid curves depict $m_1 = 2\Theta(1 - m_2)$ (left) and $m_1 = 2\Theta(\sqrt{m_2} - m_2)$ (right). The thick dashed curves are similar but for the upper bound (13). The shaded regions indicate the region of parameters for which each theorem provides structures which attain both bounds.

On the right of Figure 11, similar information is depicted for Theorem 6. In particular, the thick solid curve represents $m_1 = 2\Theta(\sqrt{m_2} - m_2)$. (Note the endpoints $m_2 = 1, m_1 = m_3 = 0$ and $m_2 = m_1 = 0, m_3 = 1$.) The dashed line represents the similar curve for the upper bound. The figure illustrates quite clearly how powerful Theorem 6 is for small $m_2$ (that is, for points near the line $m_1 + m_3 = 1$). In this case, both bounds can be proved optimal for a range of volume fractions including those where $m_1$ or $m_3$ are very close to zero.

We should remark that the corresponding figure for Theorem 2 coincides with the right side of Figure 11. This is because Theorem 2 and Theorem 6 are identical for isotropic structures. However, Theorem 2 does not apply to anisotropic structures, while Theorem 6 certainly does. Similar figures to Figure 11 can be produced for varying degrees of anisotropy. In particular, if the ratio of the eigenvalues of $K^*$ is nearly one, Figure 11 changes only slightly. Theorem 6 can be considered a generalization of Theorem 2 to anisotropic structures.

An inner bound of the G-closure. From Theorem 6 and the results of Cherkaev and Gibiansky (1996), we can produce a naive inner bound of the G-closure by lamination. Consider Figure 12a. Here we have plotted an inner and
outer bound of the G-closure for the parameters

\[ k_1 = 1, \quad k_2 = 2, \quad k_3 = 5, \]
\[ m_1 = 0.104, \quad m_2 = 0.5, \quad m_3 = 0.396. \]

The union of the unshaded and shaded region is an outer bound for the G-closure formed from the Wiener and translation bounds. In this case, \( m_1 \) is very close to \( 2\Theta(\sqrt{m_2} - m_2) \) so that case (ii) of Theorem 6 applies.

We mark the most anisotropic effective tensor given by this theorem by \( T \). We also mark the least anisotropic effective tensor on the Wiener bound (Cherkaev and Gibiansky (1996)) by \( H \). By laminating the structures with effective tensors \( T \) and \( H \), we form a family of structures which lie on the uppermost curve connecting the two points. Since this curve necessarily lies in the G-closure for the given parameters, we find that the unshaded region of the figure depicts an inner bound on the G-closure. The shaded region shows what is still not known for these parameters. The bounds do not determine whether these points belong to the G-closure or not. We remark that there is a similar region of “unknown points” near the point marked \( U \) in the Figure. However, for these parameters and at the scale of the figure, the region is impossible to see.

![Figure 12](image-url)

**Figure 12.** Inner and outer bounds on the G-closure for two different sets of admissible parameters. The bounds do not determine whether the points lying in the shaded regions are a part of the G-closure or not.

Figure 12b is similar. We use the same conductivity parameters, but less “extreme” values of the volume fractions:

\[ m_1 = 0.25, \quad m_2 = 0.5, \quad m_3 = 0.25. \]

There are very small (compared to the area of the G-closure) regions of unknown points near the “corners” where the bounds intersect. That is, \( T \)
and $H$ are very close to each other. In this case, both the inner and outer bounds are very close to the G-closure itself.

If we allow $m_1$ to decrease far enough, eventually this construction will not work. When $m_1 = 2\Theta(\sqrt{m_2 - m_2})$, the points $T$ and $T'$ of Figure 12a coincide. For $m_1 < 2\Theta(\sqrt{m_2 - m_2})$, we know of no structures which attain the lower translation bound. In this case, a simple bound can be obtained by laminating the points $H$ and $H'$ (that is, create a polycrystal of the anisotropic material $H$). In all cases, a better inner bound could be obtained by more carefully mixing known optimal points. This is a difficult problem, however, and we do not discuss it further in this paper.

5. Fields in optimal structures

5.1. Local fields required by the translation bound. Starting the analysis of the fields in optimal structures, we first revisit the translation bound (see Hashin and Shtrikman (1962); Lurie and Cherepov (1984); Tartar (1979, 1985)). Here, we sketch the ideas of the derivation, focusing on the conditions on the fields inside the material of optimal structures.

The lower bound uses the quasi-affineness of the determinant function

$$\int_{\Omega} \det DU \, dx = \det E, \quad \forall E \in \mathbb{R}^{2\times2}, \forall U \in H_1^1(\Omega)^2 + Ex$$

which we can verify for smooth functions by writing

$$\det DU = \text{div} \left( u_1 \left( \frac{\partial u_2}{\partial y}, -\frac{\partial u_2}{\partial x} \right) \right)$$

and applying the Divergence Theorem. The general result follows by approximation. The construction of the lower bound is as follows. We begin by adding and subtracting the constant $2t \det E$ for some $t \in \mathbb{R}$.

$$W(K, E) = \inf_{U \in H_1^1(\Omega)^2 + Ex} \int_{\Omega} \left( \langle DU, K \rangle + 2t \det DU \right) \, dx - 2t \det E.$$ 

Next, we relax the differential constraint on the field $DU$ by replacing the set $H_1^1(\Omega)^2 + Ex$ with the set of $F \in L^2(\Omega; \mathbb{R}^{2\times2})$ such that $\int_{\Omega} F \, dx = E$. Thus, we have

$$W(K, E) \geq \inf_F \int_{\Omega} \left( \langle F, K \rangle + 2t \det F \right) \, dx - 2t \det E$$

such that $\int_{\Omega} F \, dx = E$.

This equation gives a family of bounds on $W(K, E)$ parameterized by $t$. Here we consider the case $t = \pm k_1$. We will choose $t = k_1$ for the rest of this section. The other case is analogous. It is not guaranteed that a minimizer, $F$, of the right-hand side of (31) will be a gradient, but if it is (or if it can be approximated in the appropriate sense by a sequence of gradients) then the translation bound is optimal. In the rest of the section, we analyze the conditions of a minimizer, $F$ (with $t = k_1$). Then we construct gradient
fields $DU \in H^1_0(\Omega)^2 + Ex$ which approximate the minimizer $F$, proving that the bound is optimal in the cases we discussed in Section 4.

To simplify the calculation, we use a rotation-invariant decomposition of the quadratic forms on two-by-two matrices defined by $Q_1(F) = |F|^2 = \langle F, F \rangle$ and $Q_2(F) = 2 \det F$ (see Astala and Miettinen (1998)). Namely, we take the zero sets $\mathcal{H}^+ = (Q_1 - Q_2)^{-1}(0)$ and $\mathcal{H}^- = (Q_1 + Q_2)^{-1}(0)$. Then $\mathbb{R}^{2\times2} = \mathcal{H}^+ \oplus \mathcal{H}^-$

where

$\mathcal{H}^+ = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}, \quad \mathcal{H}^- = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} : a, b \in \mathbb{R} \right\}$.

In particular, we can write

$F = F^+ + F^-, \quad F^+ \in \mathcal{H}^+, \quad F^- \in \mathcal{H}^-$

where

$F^+ = \frac{1}{2}(F + \text{cof} F), \quad F^- = \frac{1}{2}(F - \text{cof} F)$.

Here cof is the linear operator on matrices which returns the cofactor matrix.

$\text{cof} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} f_{22} & -f_{21} \\ -f_{12} & f_{11} \end{pmatrix}$.

It is easy to verify that

$|F^+|^2 + |F^-|^2 = |F|^2, \quad |F^+|^2 - |F^-|^2 = 2 \det F$.

Differentiating the integrand in (31) with respect to $F$ we find the conditions for a minimizer:

$F(x)K(x) + k_1 \text{cof} F(x) = A, \quad \text{a.e. in } \Omega$

or, equivalently,

$F^+(x)(K(x) + K_1) + F^-(x)(K(x) - K_1) = A, \quad \text{a.e. in } \Omega$

where $A$ is a constant matrix of Lagrange multipliers enforcing (32). In particular, we find that in $\Omega_1$ (where $K \equiv K_1$) we have

$2k_1 F^+ = A \quad \text{a.e. in } \Omega_1$

which implies that $A^- = 0$ in $\Omega$. For $\Omega_i$ with $i = 2, \ldots, N$, we then have

$(k_i + k_1) F^+ = A, \quad F^- = 0 \quad \text{a.e. in } \Omega_i$.

Using (36) and (37) we solve for $A$ by noting that

$E = \int_{\Omega} F \, dx = A \sum_{i=1}^N \frac{m_i}{k_i + k_1} + \int_{\Omega_1} F^- \, dx$

so that

$A = \left( \sum_{i=1}^N \frac{m_i}{k_i + k_1} \right)^{-1} E^+, \quad \int_{\Omega_1} F^- \, dx = E^-$. 

Thus, we have the following theorem.

**Theorem 7.** A vector field $F \in L^2(\Omega; \mathbb{R}^{2 \times 2})$ is a minimizer of the right-hand-side of (31) if and only if

1. $F^+ = \frac{1}{k_i + k_1} \left( \sum_{j=1}^{N} \frac{m_j}{k_j + k_1} \right)^{-1} E^+$ a.e. in $\Omega_i$ for $i = 1, \ldots, N$.
2. $\int_{\Omega_1} F^- \, dx = E^-.$
3. $F^- = 0$ a.e. in $\Omega_i$ for $i = 2, \ldots, N$.

In particular, we can explicitly write the lower bound as

$$W(K, E) \geq \left( \sum_{i=1}^{N} \frac{m_i}{k_i + k_1} \right)^{-1} |E^+|^2 - 2k_1 \det E, \quad \forall E \in \mathbb{R}^{2 \times 2}.$$  

The traditional form of the translation bound (12) is obtained by choosing $E$ to make the bound above as tight as possible, and thus eliminating the dependence of the bound on the fields. However, we are more interested in Theorem 7 since it tells us exactly what the fields in each material of an optimal structure are.

Specifically, notice that the value of $F$ is constant with $F^- \equiv 0$ in all but the first material: $\Omega_2, \ldots, \Omega_N$. Furthermore, $F^+$ is fixed in the remaining material $\Omega_1$. The only “freedom” we have in our choice of $F$ is the values of $F^-$ in $\Omega_1$ which are arbitrary as long as we satisfy the constraint on the average, (ii). If the translation bound is optimal, then the corresponding structures contain pointwise fields which are (nearly) constant in $\Omega_2, \ldots, \Omega_N$ and constrained to belong to the two-dimensional manifold

$$\frac{1}{2k_1} \left( \sum_{j=1}^{N} \frac{m_j}{k_j + k_1} \right)^{-1} E^+ + H^-$$

in $\Omega_1$. Here $H^-$ is defined in (33).

**Remark 8.** The problem of finding optimal structures has thus been reduced to a differential inclusion problem: find $U$ with a given average such that

$$DU \in \mathcal{K} \subset \mathbb{R}^{2 \times 2} \quad a.e. \text{ in } \Omega.$$

5.2. Rank-one connection and the fields in the optimal structures.

As in Section 4, we deal with the class of orthogonal laminates — that is, the class of laminates with mutually orthogonal normals. The effective properties of these structures are found by iterating the equation (14) assuming that the normal is equal either to $n_1 = (1, 0)$ or to $n_2 = (0, 1)$. It is enough to consider only diagonal average fields (7) (see for example Milton (2002)) so that

$$E^+ = \alpha I, \quad \text{and} \quad A = \left( \sum_{i=1}^{N} \frac{m_i}{k_i + k_1} \right)^{-1} \alpha I$$

(39)
for some $\alpha \in \mathbb{R}$.

**Remark 9.** We choose laminates because the fields can be taken to be constant in each layer, making them easy to work with. In actuality, there can be fluctuations, especially near the boundaries. But with well-separated scales (see Briane (1994)) tending toward length zero, the fields may be assumed constant for the purpose of computing the energy $\mathcal{W}(K,E)$. Technically speaking, what we will continue to call “fields” are called “correctors” in the language of homogenization. The most elegant mathematical formulation is given through the use of the so-called gradient Young measures. The interested reader is referred to Müller (1999) for an introduction to the subject in the context of materials science.

The fields of neighboring layers in a laminate structure are rank-one connected. In other words, two neighboring layers with respective fields $F_A$ and $F_B$ must satisfy

$$\det(F_A - F_B) = 0.$$  

On the other hand, the fields in each material in an optimal structure are given by Theorem 7 with $F = DU$. We now analyze these conditions, showing how they guide us in creating optimal structures.

The magnitude of the average field does not affect the effective properties, therefore it is convenient to rescale the fields in the theorem so that

$$\alpha = k_1 \sum_{i=1}^{N} \frac{m_i}{k_i + k_1}, \quad E^+ = \alpha I, \quad A^+ = k_1 I$$

in (39). Thus, in an optimal structure, the field $DU$ satisfies

$$DU^+ = \frac{K_1}{k_i + k_1} I \quad \text{a.e. in } \Omega_i \text{ for } i = 1, \ldots, N.$$ 

Since $\text{tr} F = \text{tr} F^+$, we can rephrase the conditions of Theorem 7 as follows.

(P1) $DU = DU^T$ and $\text{tr} DU = 1$ a.e. in $\Omega_1$.

(P2) $\int_{\Omega_1} DU^- dx = E^-.$

(P3) $DU = \frac{K_1}{k_i + k_1} I \quad \text{a.e. in } \Omega_i \text{ for } i = 2, \ldots, N.$

**Remark 10.** We refer the reader to Grabovsky (1996) for a detailed derivation of similar conditions for composites of two generally anisotropic linear elastic materials. The isotropic case is covered in (3.26) and (3.27) of his paper which the reader may wish to compare to (P1)–(P3) above with $N = 2$. The same paper also makes the connection between optimal composites in linear conductivity and in special cases of linear elasticity. We further remark that conditions (P1)–(P3) are a special case of (25.31) in Milton (2002).

One immediately observes that the fields in all materials except $K_1$ are not in rank-one connection with each other and therefore are incompatible, because (P1)–(P3) and (40) are contradictory: if $F_A = \frac{k_1}{k_2 + k_1} I$ and $F_B =$
In a sense, $K_1$ must be used as a glue between layers to ensure compatibility. The volume fraction of $K_1$ therefore cannot be too small, which indicates that the translation bound (12) cannot be optimal for laminates if $m_1$ is smaller than a critical value.

5.3. **Constructing optimal laminates.** We now describe an algorithm for constructing laminates which are optimal for (12). We always assume that the fields in the layers of the laminate satisfy (P1)–(P3) and we find ways to join the materials by rank-one connection into laminate structures. Since we require that (P1)–(P3) hold at all times in the process, the final structures are necessarily optimal.

We leave the average field, $E$, and the volume fractions free until the end of the process, when they are computed from the construction. In other words, by following the procedure described in this section, we are guaranteed to produce optimal structures, but we do not know the parameters of the optimization problem (the average field and volume fractions) until the structure is complete. However, at the end of the process, we have an algorithm to find all G-closure parameters.

5.4. **The general structure revisited.** Consider again the general structure illustrated in Figure 6. Notice that there are six design parameters (the volume fractions in the laminate layers). Since the structure is an orthogonal laminate and since we assume the average field $E$ is diagonal, we have that the local fields are also diagonal. Notice that material $K_2$ appears in lamination only once, material $K_3$ twice, and material $K_1$ four times. Thus, we have 6 design variables and 14 field variables. To make the structure optimal, we must fill in the fields in the pure material components satisfying a number of conditions. Let us count these conditions.

First, we have rank-one connections between the fields and the currents in each layer of the laminate. This gives two condition per layer, since the tangential component of the field and the normal component of the current must be continuous across the interfaces. There are six laminate interfaces, so we have 12 continuity conditions.

For optimality, we also need to satisfy (P1)–(P3). These give two conditions for each layer of $K_2$ and $K_3$ and one condition for each layer of $K_1$. Thus, (P1)–(P3) impose 10 more conditions.

By our count, we have 20 free variables and 22 conditions they must satisfy. In fact, we observe that the general structure is a coated $T^2$-structure. The $T^2$-structure has two free design parameters ($\omega_1$ and $\omega_2$). The coating introduces two more parameters, so we actually have four degrees of freedom in spite of the seemingly over-determined system. As we will show in the following sections, some of the constraints are satisfied “for free” if the others are satisfied. In particular, when we choose parameters to satisfy the rank-one connections and (P1)–(P3) for the fields in the general
structure, the currents are automatically rank-one connected. This removes six constraints from the list above and we are left with 20 variables and 16 constraints, providing the four degrees of freedom we observe.

This is not as surprising as it may at first seem. Indeed, assume we have a partition of \( \Omega \) into \( \Omega_i \) with associated conductivity tensor \( K \) defined through (6). Furthermore, assume that we find \( U \in H^1_2(\Omega)^2 + Ex \) that satisfies the optimality condition (35) with \( F = DU \). Then, by taking the divergence of both sides of (35) and using the fact that \( \text{div}(\text{cof} \ DU) = 0 \), we find that \( U \) satisfies the PDE \( \text{div}(K DU^T) = 0 \) automatically. The analogous statement for laminates is that it is enough to check the jump conditions only of the piecewise constant field approximating \( DU \). We obtain “for free” the corresponding conditions for the piecewise constant field approximating \( DU K \).

The optimal T-structures. The construction of optimal laminates preserves the fields in the layers according to (P1)–(P3). By restricting ourselves to orthogonal laminates and diagonal average fields, we guarantee that the fields in each layer will be diagonal, which allows us to simplify notation. We represent the diagonal two-by-two matrices as

\[
M(\alpha, \beta) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}
\]

and associate \( M(\alpha, \beta) \) with the point \((\alpha, \beta)\) in the plane. Observe that \( M(\alpha_1, \beta_1) \) and \( M(\alpha_2, \beta_2) \) are rank-one connected if and only if \((\alpha_2 - \alpha_1)(\beta_2 - \beta_1) = 0\). In the plane, this means that \((\alpha_1, \beta_1)\) and \((\alpha_2, \beta_2)\) lie on the same horizontal or vertical line.

We now prove the optimality of the T-structure. The discussion is accompanied by Figure 13a. In this figure, the points \( E_2 \) and \( E_3 \) represent the fixed fields given by (P3). The line \( l \) represents the line of constant trace given by (P1). We wish to construct a laminate which has its internal fields lying on the set \( \{E_2\} \cup \{E_3\} \cup l \).

First, we look for a rank-one connection between the materials \( K_1 \) and \( K_3 \). The admissible fields for \( K_1 \) lie on the line \( l \) while the admissible field for \( K_3 \) is the point \( E_3 \). Let us laminate in the \( x_1 \)-direction, which means the field in \( K_1 \) must lie on the intersection of \( l \) and the horizontal line through the fixed field \( E_3 \). The optimal fields \( E_1 \) and \( E_3 \) are

\[
E_1 = M \left( \frac{k_3}{k_3 + k_1}, \frac{k_1}{k_3 + k_1} \right), \quad E_3 = M \left( \frac{k_1}{k_3 + k_1}, \frac{k_1}{k_3 + k_1} \right),
\]

which ensures that

\[
(E_1 - E_3) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{k_1}{k_3 + k_1} - \frac{k_1}{k_3 + k_1} = 0.
\]

As we mentioned in Section 5.4, the rank-one connection condition on the currents is automatic. So far, we have only ensured the condition on the
fields. However, note that
\[(k_1 E_1 - k_3 E_3) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = k_1 k_3 / (k_3 + k_1) - k_1 k_3 / (k_3 + k_1) = 0\]
so the currents are also rank-one connected.

The average field upon laminating $K_1$ with $K_3$ is
\[E_{13} = \nu E_1 + (1 - \nu) E_3 = M \left( \frac{\nu k_3 + (1 - \nu) k_1}{k_3 + k_1}, \frac{k_1}{k_3 + k_1} \right)\]
while the average current is
\[J_{13} = \nu k_1 E_1 + (1 - \nu) k_3 E_3 = M \left( \frac{k_1 k_3}{k_3 + k_1}, \frac{\nu k_1^2 + (1 - \nu) k_1 k_3}{k_3 + k_1} \right)\]
where $\nu$ is the relative fraction of $K_1$ to $K_3$. In Figure 13a, we indicate the point $E_1$. The dashed line connecting this point to $E_3$ represents the path of the point $E_{13}$ as $\nu$ varies between 0 and 1. All of these fields are available to us in an optimal laminate of $K_1$ and $K_3$ by appropriate choice of $\nu$.

We choose $\nu$ so that we can laminate this new material with $K_2$ in the $x_2$-direction (Refer to Figure 7a.), again satisfying (P1)–(P3). Thus, we need to adjust $\nu$ so that $E_{13}$ and $E_2$ lie on the same vertical line in the plane, where $E_2 = M \left( \frac{k_1}{k_2 + k_1}, \frac{k_1}{k_2 + k_1} \right)$. Solving for $\nu$, we find
\[\nu = \frac{k_1 (k_3 - k_2)}{(k_2 + k_1)(k_3 - k_1)} = \Theta.\]
Note that $\Theta$ is defined in (17) and is a parameter of the optimal structures in the previous section.

Once again, we can verify that the currents in the laminate layers are compatible. When $\nu = \Theta$,
\[J_{13} = M \left( \frac{k_1 k_3}{k_3 + k_1}, \frac{k_1 k_2}{k_2 + k_1} \right)\]
which is rank-one connected to the current $k_2 E_2$. Similar calculations for all other structures discussed in this paper show that the condition on currents is automatically satisfied each time we satisfy the condition on the fields, so for the rest of this section, we will keep track only of the fields and not the currents in the construction.

Setting $\nu = \Theta$ brings the point $E_{13}$ to the vertical line passing through $E_2$ and we are free to laminate with as much $K_2$ as we please. The average field in the T-structure can lie anywhere on the dashed line connecting $E_{13}$ and $E_2$. The only condition we have is the ratio of $m_1$ to $m_3$, leaving $m_2$ free. In this way, the average field, $E$, depends on $m_2$. We then find the volume fractions of the optimal T-structure depending on the amount $K_2$ parameterized by $\omega_1$:

$$m_1 = \Theta(1 - \omega_1), \quad m_2 = \omega_1, \quad m_3 = (1 - \Theta)(1 - \omega_1).$$

To find the effective properties, we need to find the average field,

$$E_T = m_1 E_1 + m_2 E_2 + m_3 E_3$$

and the average current,

$$J_T = m_1 k_1 E_1 + m_2 k_2 E_2 + m_3 k_3 E_3$$

Then we have that $K^* = E_T^{-1} J_T$. Substituting and simplifying, we get the eigenvalues of $K^*$:

$$\lambda_1 = \frac{m_2 k_2 (k_3 + k_1) + (1 - m_2) k_3 (k_2 + k_1)}{k_3 + k_1}$$

(44)

$$\lambda_2 = \frac{k_2 (k_3 + k_1)}{m_2 (k_3 + k_1) + (1 - m_2)(k_2 + k_1)}.$$

These expressions coincide with (29) and (30) for $\omega_1 = m_2, \omega_2 = 1$. We can now prove Theorem 3 by simple algebra using $m_2 = p(m_2 + m_3)$, (42), (44) and (45).

The coating principle. It is now easy to see why the coating principle (Theorem 4) is true. Consider Figure 13b. Starting from the optimal T-structure, $E_T$, we look for compatible fields for lamination. Notice that since the field in $K_1$ can lie anywhere on the line $l$, there are always two compatible fields: the intersection of the horizontal or vertical line through $E_T$ and the line $l$. For this illustration, we have chosen the point $E_1' \in l$ which lies on the same horizontal line as $E_T$. Specifically, from (43) we find

$$E_1' = \frac{1}{(k_3 + k_1)(k_2 + k_1)} M(\omega_1 k_2 (k_3 + k_1) + (1 - \omega_1) k_3 (k_2 + k_1),$$

$$\omega_1 (k_3 + k_1) + (1 - \omega_1)(k_2 + k_1)).$$
By varying the amount of $K_1$ which is added in this layer, we can obtain a new optimal structure $E_{CT}$ which lies anywhere on the dashed line connecting $E_T$ and $E'_1$. (Refer to Figure 7b.) Of course, this operation can be iterated. For example, we could then laminate in the $x_2$-direction with $K_1$ by choosing the point on $l$ which intersects the vertical line through $K_{CT}$ and so on. In this way, we obtain a whole family of optimal structures from a single optimal structure. Again, the coating will change the average field, $E$, and volume fractions. However, the construction must give an optimal structure for some values of these parameters which we can compute.

The optimal $T^2$-structures. Finally, we illustrate how to obtain the optimal $T^2$-structures. Begin with the optimal $T$-structure, indicated by $E_T$ in Figure 13b. Notice that we can laminate $E_3$ with the point $E''_1 = M\left(\frac{k_3}{k_3+k_1}, \frac{k_3}{k_3+k_1}\right)$. By adjusting the volume fraction $\nu'$ of $K_1$ to $K_3$ in this laminate, we can move the resulting average field

$$E'_{13} = \nu'E_1 + (1-\nu')E_3 = M\left(\frac{k_1}{k_3+k_1}, \frac{\nu'k_3 + (1-\nu')k_1}{k_3+k_1}\right)$$

to any point on the dashed line connecting $E_3$ and $E''_1$. In particular, if we choose the volume fraction so that $E'_{13}$ is rank-one connected to $E_T$ — that is if we choose $\nu'$ so that

$$\frac{\nu'k_3 + (1-\nu')k_1}{k_3+k_1} = \frac{\omega_1(k_3 + k_1) + (1-\omega_1)(k_2 + k_1)}{k_3+k_1}$$

(see (43)) — then we can laminate the $E'_{13}$ and $E_T$ structures to obtain a $T^2$-structure (see Figure 7c). It is easy to check that the correct value of $\nu'$ is $\nu' = \omega_1\Theta$ with $\Theta$ defined in (17). We are then free to laminate $E_T$ with $E'_{13}$ in any fraction we wish to obtain an optimal $T^2$-structure:

$$E_{T2} = \omega_2E_T + (1-\omega_2)E'_{13}.$$ 

By computing the volume fractions, average field, and average current as we did with the $T$-structure, one can verify all the statements of Theorem 5. We can now apply the coating principle as many times as we wish to obtain a family of optimal structures as described in Section 4.

### 5.5. Generalizations.

Four and more materials. The generalization to $N \geq 4$ is straightforward. In fact, Theorem 7 was proven in the general case. The method for constructing optimal laminates is exactly the same: we are given a set $K$ of $N-1$ points and a two-dimensional plane in $\mathbb{R}^{2\times2}$ and we construct laminate structures whose internal fields are lie in $K$. As an example, Figure 14a shows the rank-one construction of a four-material generalization of the T-structure and Figure 14b shows the associated laminate structure. Laminates of $K_1$ and $K_3$ (and $K_1$ and $K_4$) are brought into rank-one connection with material $K_2$ respecting (P1)–(P3), allowing for second-rank lamination. The improvements of the applicability conditions for the method discussed in this paper over previous results become more pronounced with larger $N$. 
6. AN AUXILIARY BOUND

In Section 5, we showed that a structure is optimal for the bound (12) provided certain conditions (P1)–(P3) hold in an approximate sense. Furthermore, we constructed optimal structures exactly by satisfying these conditions. The present section addresses a new issue. Do there exist structures optimal for (12) which are not equivalent to those we have found? (Here we will consider two structures equivalent if they have the same volume fractions \( m = (m_1, \ldots, m_N) \) and generate the same \( K^* \in G(m; k) \).) An answer is not yet known for general \( K^* \in G(m; k) \). However, in this section, we prove an auxiliary bound that any effective tensor, \( K_{\text{eff}} \), must satisfy if it attains the translation bound, provided the interfaces between materials is smooth. The auxiliary bound limits the anisotropy of such a structure and has an interesting relationship to the \( T^2 \)-structures discussed in Section 4. For this section, we assume the average field \( E \) has been rescaled as in (41).

6.1. The bound.

**Theorem 8.** Let \( k = (k_1, k_2, k_3) \) and \( m = (m_1, m_2, m_3) \) be admissible parameters. Define

\[
\gamma = \sum_{i=1}^{3} \frac{m_i}{k_i + k_1}, \quad \delta = \frac{m_1 k_3}{k_1 (k_3 + k_1)^2} + \frac{m_2}{(k_2 + k_1)^2} + \frac{m_3}{(k_3 + k_1)^2}.
\]

Assume that there exists three open sets \( \Omega_1, \Omega_2, \Omega_3 \) with smooth boundaries such that \( |\Omega_i| = m_i \) and \( \bigcup_{i=1}^{3} \Omega_i = \Omega = [0, 1]^2 \) and assume that there exists a field \( U \in H^1_{\#}(\Omega)^2 + Ex \) which satisfies conditions (i)–(iii) in Theorem 7 for
\( F = DU \) and for average field \( E \in \mathbb{R}^{2 \times 2} \) such that
\[
E^+ = k_1 \left( \sum_{i=1}^{3} \frac{m_i}{k_i + k_1} \right) I = k_1 \gamma I.
\]
Let \( \lambda_1 \) and \( \lambda_2 \) be the eigenvalues of the effective tensor \( K_{\text{eff}} \). Then one has
\[
(47) \quad \left( \frac{\lambda_1 - \lambda_2 \lambda_1 \lambda_2 - k_1^2}{\lambda_1 \lambda_2 - k_1^2} \right)^2 \leq 4(\gamma^2 - \delta).
\]

**Remark 11.** To better understand the nature of (47), suppose we have labeled the eigenvalues so that \( 0 < \lambda_1 \leq \lambda_2 \) and that the right-hand side of (47) is nonnegative. Since we assume that \( K_{\text{eff}} \) is optimal, the eigenvalues must also satisfy (see (12))
\[
\frac{\lambda_1 + \lambda_2 - 2k_1}{\lambda_1 \lambda_2 - k_1^2} = 2\gamma.
\]
Using this together with the inequality (47), one finds an upper bound on \( \lambda_2 \) in terms of \( \lambda_1 \):
\[
(\gamma - \sqrt{\gamma^2 - \delta}) \lambda_2 \leq (\gamma + \sqrt{\gamma^2 - \delta}) \lambda_1 - 2k_1 \sqrt{\gamma^2 - \delta}.
\]

Before proving Theorem 8, we need a bound on the pointwise fields in the structure

**Proposition 1.** Under the hypotheses of Theorem 8, \( \det DU \) satisfies the inequality
\[
(48) \quad \det DU \geq \frac{k_1 k_3}{(k_3 + k_1)^2} \quad \text{in } \Omega_1.
\]

**Proof.** Observe that (i)–(iii) in Theorem 7 implies that
\[
(49) \quad \det DU = \frac{1}{4} - \frac{1}{2} |DU^-|^2 \quad \text{in } \Omega_1,
\]
\[
\det DU = \frac{k_1^2}{(k_i + k_1)^2} \quad \text{in } \Omega_i \text{ for } i = 2, 3.
\]
Write the vector function \( U \) in terms of its components \( U = (u, v) \). Then
\[
|DU^-|^2 = \frac{1}{2} (u_x - v_y)^2 + \frac{1}{2} (u_y + v_x)^2.
\]
Let \( w = u_x - v_y \) and \( z = u_y + v_x \). Then since \( u \) and \( v \) are harmonic in \( \Omega_1 \), so are \( w \) and \( z \). We now show that \( - \det DU \) is subharmonic in \( \Omega_1 \). Indeed,
\[
\Delta(-\det DU) = \frac{1}{2} \Delta |DU^-|^2
\]
\[
= \frac{1}{2} \Delta (w^2 + z^2)
\]
\[
= w \Delta w + |\nabla w|^2 + z \Delta z + |\nabla z|^2
\]
\[
= |\nabla w|^2 + |\nabla z|^2 \geq 0.
\]
Therefore, \( \det DU \) is smooth in \( \Omega_1 \) and satisfies the strong minimum principle there.

On the other hand, the assumption of smooth boundaries implies that the vector potential satisfies the transmission conditions. If \( t \) is the tangent to the boundary at a point and \( n \) is the normal, we have \( DU \cdot t \) and \( DU \cdot n \) are continuous across an interface. As a consequence of the invariance under rotations of the determinant function we get that the function \( K \det DU \) must be continuous across the interfaces.

Using the fact that \( DU \) is constant on \( \Omega_2 \) and \( \Omega_3 \) along with the continuity of \( K \det DU \) and the minimum principle, we find that

\[
\det DU \geq \min \left\{ \frac{k_1 k_2}{(k_2 + k_1)^2}, \frac{k_1 k_3}{(k_3 + k_1)^2} \right\}
\]

\[
= \frac{k_1 k_3}{(k_3 + k_1)^2} \quad \text{in } \Omega_1.
\]

\( \square \)

**Proof of Theorem 8.** Begin with (35) which implies

\[
DU(x)K(x) + k_1 \cof DU(x) = A = k_1 I, \quad \text{in } \Omega.
\]

Integrating this equation on \( \Omega \), we have (by the definition of \( K_{\text{eff}} \))

\[
EK_{\text{eff}} + k_1 \cof E = k_1 I
\]

which we can solve for

\[
E = k_1 \frac{\cof K_{\text{eff}} - k_1 I}{\det K_{\text{eff}} - k_1^2}.
\]

Assume that the material has been oriented in such a way that \( K_{\text{eff}} = \text{diag}(\lambda_1, \lambda_2) \) for \( \lambda_1, \lambda_2 > 0 \). Then we find

\[
E^+ = \frac{1}{2} (E + \cof E) = k_1 \frac{1}{2} \left( \frac{\lambda_1 + \lambda_2 - 2k_1 \lambda_1 \lambda_2 - k_1^2}{\lambda_1^2 \lambda_2 - k_1^2} \right) I,
\]

\[
E^- = \frac{1}{2} (E - \cof E) = k_1 \frac{1}{2} \left( \frac{\lambda_2 - \lambda_1}{\lambda_1^2 \lambda_2 - k_1^2} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).
\]

Note that \( E^+ = k_1 \gamma I \) and the above equation for \( E^+ \) are consistent since we assume that the translation bound is attained. That is,

\[
E^+ = k_1 \gamma I = \frac{k_1}{2} \left( \frac{\lambda_1 + \lambda_2 - 2k_1}{\lambda_1 \lambda_2 - k_1^2} \right) I \Longleftrightarrow \frac{\lambda_1 + \lambda_2 - 2k_1}{\lambda_1 \lambda_2 - k_1^2} = 2 \sum_{i=1}^{3} \frac{m_i}{k_i + k_1}.
\]

To obtain the desired inequality, consider

\[
(50) \quad 2 \det E = |E^+|^2 - |E^-|^2 = 2k_1^2 \gamma^2 - \frac{k_1^2}{2} \left( \frac{\lambda_2 - \lambda_1}{\lambda_1 \lambda_2 - k_1^2} \right)^2.
\]

Using the quasi-affineness of the determinant, we also have

\[
2 \det E = \int_{\Omega} 2 \det DU(x) \, dx = \int_{\Omega_1} 2 \det DU(x) \, dx + \int_{\Omega \setminus \Omega_1} 2 \det DU(x) \, dx.
\]
We know $\det DU$ in $\Omega \setminus \Omega_1$ and we use (48) in $\Omega_1$ to obtain
\[
2 \det E \geq \frac{2m_1 k_1 k_3}{(k_3 + k_1)^2} + \frac{2m_2 k_2^2}{(2k_2 + k_1)^2} + \frac{m_3 k_1^2}{(k_3 + k_1)^2} = 2k_1^2 \delta.
\]
Combining this with (50), we obtain (47).

6.2. Discussion. Theorem 8 requires that any “exact structure” with smooth interfaces satisfy (47). One can check that the right-hand side of the inequality vanishes when $m_1 = 2\Theta(\sqrt{m_2 - m_2})$ and is negative for $0 \leq m_1 < 2\Theta(\sqrt{m_2 - m_2})$. Thus no structure with smooth interfaces can attain the bound for this range of volume fractions. The validity of the auxiliary bound for arbitrary structures will be addressed elsewhere. However, observe that this critical relationship of $m_1 = 2\Theta(\sqrt{m_2 - m_2})$ is exactly the relationship which holds between the volume fractions of the most extreme version of the isotropic structures introduced in Gibiansky and Sigmund (2000). (This structure is represented by the point $K_{GS}$ in Figure 9.) It is also interesting to note that if we formally set $K_{eff} = \text{diag}(\lambda_1, \lambda_2)$ with $\lambda_1, \lambda_2$ given in (29),(30) and set the volume fractions accordingly, then (47) holds as equality. Thus the $T^2$-structures — which have the smallest values of $m_1$ among all known optimal structures — while not exact, satisfy the auxiliary bound (47) as equality. For all other structures in this paper which are optimal for (12), the inequality is strict.

Appendix A

Here we show the difference between the concepts of optimality and attainability of an outer bound on the G-closure, as defined in Section 3.2. In particular, we exhibit points on the Wiener bounds (10) and (11) which are optimal but not attainable. First we recall the results published in Cherkaev and Gibiansky (1996). These results can be summarized by the following theorem.

**Theorem 9 (Cherkaev-Gibiansky).** Let
\[
0 < k_1 < k_2 < k_3 \quad \text{and} \quad m_1, m_2, m_3 > 0
\]
such that $m_1 + m_2 + m_3 = 1$ be admissible parameters to the G-closure problem. Define
\[
\overline{k} = \sum_{i=1}^3 m_i k_i, \quad \underline{k} = \left(\sum_{i=1}^3 \frac{m_i}{k_i}\right)^{-1}.
\]

Then there exist numbers $\alpha$ and $\beta$ below) such that
\[
\alpha < \overline{k} \quad \text{and} \quad \underline{k} < \beta
\]
and such that...
(i) The points on the closed line segment joining the points \((k, k)\) and \((k, \alpha)\) are optimal for the Wiener bound (10).

(ii) The points on the closed line segment joining the points \((\beta, k)\) and \((\beta, \bar{k})\) are optimal for the Wiener bound (11).

Proof. The proof uses one of the iterated laminates illustrated in Figure 15. The important parameter is the relative volume fraction, \(\nu\), of \(K_1\) to \(K_3\) in the inner laminate layer. To find \(\alpha\), one fixes \(\nu\) so that the laminate has the same conductivity in the \(x_1\) direction as \(K_2\). This ensures that when this composite is laminated with \(K_2\) in the \(x_2\) direction, the current does not jump across the interface. It is easy to check that

\[
\nu = \nu_\alpha = \frac{k_1(k_3 - k_2)}{k_2(k_3 - k_1)}.
\]

If

\[
m_1 \geq \frac{\nu_\alpha}{1 - \nu_\alpha} m_3,
\]

then the structure in Figure 15a is used, otherwise the structure in Figure 15b is used.

To find \(\beta\), one instead chooses \(\nu\) so that the conductivity of the laminate in the \(x_2\) direction is the same as \(K_2\). This ensures that when this composite is laminated with \(K_2\) in the \(x_2\) direction, the field does not jump across the interface. In this case, one can check that

\[
\nu = \nu_\beta = \frac{k_3 - k_2}{k_3 - k_1}.
\]

If

\[
m_1 \geq \frac{\nu_\beta}{1 - \nu_\beta} m_3,
\]

then the structure in Figure 15a is used, otherwise the structure in Figure 15b is used. \(\square\)

**Figure 15.** The structures from Theorem 9.

**Remark 12.** As a particular example, for the parameters of Figure 12b, one has

\[
\nu_\alpha = \frac{3}{8} \quad \text{and} \quad \nu_\beta = \frac{3}{4}.
\]
One finds that
\[ m_1 > \frac{\nu_\alpha}{1 - \nu_\alpha} m_3 = \frac{3}{5} m_3 \quad \text{and} \quad m_1 < \frac{\nu_\beta}{1 - \nu_\beta} m_3 = 3 m_3. \]
Thus, we can calculate \( \alpha \) and \( \beta \) from the theorem as
\[ \alpha = \frac{79}{34} \approx 2.3235 < k = \frac{5}{2} = 2.5, \]
\[ \beta = \frac{170}{89} \approx 1.9101 > k = \frac{20}{11} \approx 1.8182. \]

On the other hand, we will now prove that the only attainable point on the segments described in (i) and (ii) of the theorem is the point \((\underline{k}, \overline{k})\).

**Theorem 10.** Let
\[ 0 < k_1 < k_2 < k_3 \quad \text{and} \quad m_1, m_2, m_3 > 0 \]
such that \( m_1 + m_2 + m_3 = 1 \) be admissible parameters to the G-closure problem. Define \( \underline{k} \) and \( \overline{k} \) as in the previous theorem. Then a point \( \mathbf{K}_{\text{eff}} \) is attainable for (10) or (11) if and only if \( \mathbf{K}_{\text{eff}} \) has eigenvalues \( \underline{k} \) and \( \overline{k} \).

**Proof.** The fact that \((\underline{k}, \overline{k})\) is attainable is easy; we simply use a first-rank lamination in the \( x_1 \) direction. Choose the partition
\[ \Omega_1 = (0, m_1) \times (0, 1), \]
\[ \Omega_2 = (m_1, m_1 + m_2) \times (0, 1), \]
\[ \Omega_3 = (m_1 + m_2, 1) \times (0, 1). \]
Then \( \mathbf{K}_{\text{eff}} = \text{diag}(\underline{k}, \overline{k}) \).

Now assume that there exists a partition of \( \Omega \) into \( \Omega_i \) such that for \( \mathbf{K} \) defined as in (6), the effective tensor \( \mathbf{K}_{\text{eff}} = \text{diag}(\lambda_1, \lambda_2) \) is such that \( \lambda_2 = \overline{k} \).

That is, we have
\[ \inf_{u \in H^1_0(\Omega)} \int_{\Omega} (\nabla u(x) + e_2) \cdot \mathbf{K}(x)(\nabla u(x) + e_2) \, dx = \overline{k} \]
where we denote by \( \{e_1, e_2\} \) the canonical basis in \( \mathbb{R}^2 \). Then by uniqueness of weak solutions to the PDE (5), we have \( u \equiv 0 \). It follows that for the \( \mathbf{K} \) defined through the hypothetical partition, we have
\[ \text{div}(\mathbf{K}(x)e_2) = 0 \quad \text{or} \quad \frac{\partial K(x_1, x_2)}{\partial x_2} = 0 \]
in the sense of distribution. It is well known that this implies the existence of a distribution \( \hat{K} = \hat{K}(x_1) \) such that \( K(x_1, x_2) = \hat{K}(x_1) \) as distributions.

Let us now compute the other eigenvalue \( \lambda_1 \) by considering the orthogonal applied field \( e_1 \). We solve the equation
\[ \text{div}(\mathbf{K}(x)(\nabla v(x) + e_1)) = 0 \quad \text{in} \ \Omega. \]
Define \( v \in H^1_\#(\Omega) \) such that
\[
v(x) = v(x_1), \quad \nabla v(x) = \frac{k}{K(x_1)} - e_1.
\]
Then
\[
\text{div}(K(x)(\nabla v(x) + e_1)) = \text{div} \left( \hat{K}(x_1) \cdot \frac{k}{K(x_1)} \right) = 0.
\]
By uniqueness of the solution, we have that
\[
\inf_{u \in H^1_\#(\Omega)} \int_{\Omega} (\nabla u(x) + e_1) \cdot K(x)(\nabla u(x) + e_1) \, dx
\]
\[
= \int_{\Omega} (\nabla v(x) + e_1) \cdot K(x)(\nabla v(x) + e_1) \, dx
\]
\[
= \int_{\Omega} \frac{k^2}{K(x_1)} \, dx = k^2 \sum_{i=1}^{3} \frac{m_i}{k_i} = k,
\]
which proves that \( \lambda_1 = k \). A similar argument shows that if \( \lambda_1 = \frac{k}{k} \) then \( \lambda_2 = \frac{k}{k} \).

References


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