Finite approximations of $p$-local compact groups

Alex González

Kansas State University

Workshop on Group Actions - Classical and Derived
The Fields Institute
June 13-17, 2016
Let $p$ be a prime, fixed for the rest of the talk.
Let $p$ be a prime, fixed for the rest of the talk.

**Stable Elements Theorem for finite groups (Cartan-Eilenberg):**

Let $G$ be a finite group, and let $S \in \text{Syl}_p(G)$. Then,

$$H^*(BG, \mathbb{F}_p) \cong \varprojlim_{\mathcal{F}_S(G)} H^*(-; \mathbb{F}_p) \leq H^*(BS; \mathbb{F}_p).$$
Some results for compact Lie groups

Let $p$ be a prime, fixed for the rest of the talk.

**Stable Elements Theorem for finite groups (Cartan-Eilenberg):**

Let $G$ be a finite group, and let $S \in Syl_p(G)$. Then,

$$H^*(BG, \mathbb{F}_p) \cong \varprojlim_{\mathcal{F}_S(G)} H^*(-; \mathbb{F}_p) \leq H^*(BS; \mathbb{F}_p).$$

$\mathcal{F}_S(G)$ is the category with $\text{Ob}(\mathcal{F}_S(G)) = \{P \leq S\}$ and, for $P, Q \leq S$,

$$\text{Mor}_{\mathcal{F}_S(G)}(P, Q) = \{c_x : P \to Q \mid x \in G, xPx^{-1} \leq Q\}.$$
Some results for compact Lie groups

Question:

Is there a version of the Stable Elements Theorem for compact Lie groups?
Question:
Is there a version of the Stable Elements Theorem for compact Lie groups?

Let $G$ be a compact Lie group. What is $\mathcal{F}_S(G)$ in this case?
Some results for compact Lie groups

Question:

Is there a version of the Stable Elements Theorem for compact Lie groups?

Let $G$ be a compact Lie group. What is $\mathcal{F}_S(G)$ in this case? Let $\hat{T} \leq G$ be a maximal torus, and let $W$ be the Weyl group.

$$\hat{T} \rightarrow N_G(\hat{T}) \rightarrow W$$
Some results for compact Lie groups

Question:
Is there a version of the Stable Elements Theorem for compact Lie groups?

Let $G$ be a compact Lie group. What is $\mathcal{F}_S(G)$ in this case? Let $\hat{T} \leq G$ be a maximal torus, and let $W$ be the Weyl group.

\[
\hat{T} \longrightarrow N_G(\hat{T}) \longrightarrow W
\]

Choose $\pi \in \text{Syl}_p(W)$:
Question:
Is there a version of the Stable Elements Theorem for compact Lie groups?

Let $G$ be a compact Lie group. What is $\mathcal{F}_S(G)$ in this case? Let $\hat{T} \leq G$ be a maximal torus, and let $W$ be the Weyl group.

$$
\hat{T} \rightarrow N_G(\hat{T}) \rightarrow W
$$

Choose $\pi \in \text{Syl}_p(W)$:

$$
\begin{array}{ccc}
\hat{T} & \rightarrow & N_G(\hat{T}) \rightarrow W \\
\hat{T} & \rightarrow & \hat{S} \rightarrow \pi \\
\end{array}
$$

$\hat{S}$ is a Sylow $p$-subgroup of $G$, but it is not discrete...
Question:

Is there a version of the Stable Elements Theorem for compact Lie groups?

Let $G$ be a compact Lie group. What is $\mathcal{F}_S(G)$ in this case? Let $\hat{T} \leq G$ be a maximal torus, and let $W$ be the Weyl group.

$$
\hat{T} \rightarrow N_G(\hat{T}) \rightarrow W
$$

Choose $\pi \in \text{Syl}_p(W)$:

$$
\hat{T} \rightarrow N_G(\hat{T}) \rightarrow W
\downarrow \quad \uparrow \text{incl}
\hat{T} \rightarrow \hat{S} \rightarrow \pi

\hat{S} \text{ is a Sylow } p\text{-subgroup of } G, \text{ but it is not discrete...}
Let \( T \leq \hat{T} \) be the subgroup of all \( p^n \)-th roots of 1, for \( n \geq 0 \):
Let $T \leq \hat{T}$ be the subgroup of all $p^n$-th roots of 1, for $n \geq 0$:

\[
\begin{array}{ccc}
\hat{T} & \longrightarrow & \hat{S} \\
\uparrow & & \uparrow \\
T & \longrightarrow & S
\end{array}
\]

\[
\begin{array}{ccc}
\pi & & \\
\downarrow & & \downarrow \\
\pi & & \\
\end{array}
\]

$S$ is a discrete Sylow $p$-subgroup of $G$. 

Let \( T \leq \hat{T} \) be the subgroup of all \( p^n \)-th roots of 1, for \( n \geq 0 \):

\[
\begin{array}{ccc}
\hat{T} & \longrightarrow & \hat{S} \\
\uparrow & & \uparrow \\
T & \longrightarrow & S \\
\end{array}
\]

\( S \) is a discrete Sylow \( p \)-subgroup of \( G \). Define \( \mathcal{F}_S(G) \) as the category with \( \text{Ob}(\mathcal{F}_S(G)) = \{ P \leq S \} \) and, for \( P, Q \leq S \),

\[
\text{Mor}_{\mathcal{F}_S(G)}(P, Q) = \{ c_x : P \rightarrow Q \mid x \in G, xPx^{-1} \leq Q \}.
\]
Let $T \leq \hat{T}$ be the subgroup of all $p^n$-th roots of 1, for $n \geq 0$:

\[
\begin{align*}
\hat{T} & \longrightarrow \hat{S} \longrightarrow \pi \\
\uparrow & \quad \uparrow \\
T & \longrightarrow S \longrightarrow \pi
\end{align*}
\]

$S$ is a discrete Sylow $p$-subgroup of $G$. Define $\mathcal{F}_S(G)$ as the category with $\text{Ob}(\mathcal{F}_S(G)) = \{P \leq S\}$ and, for $P, Q \leq S$,

\[
\text{Mor}_{\mathcal{F}_S(G)}(P, Q) = \{c_x : P \rightarrow Q \mid x \in G, xPx^{-1} \leq Q\}.
\]

There is a natural map

\[
H^*(BG; \mathbb{F}_p) \xrightarrow{\rho} \lim_{\mathcal{F}_S(G)} H^*(-; \mathbb{F}_p) \leq H^*(BS; \mathbb{F}_p).
\]
How can we prove that $\rho$ is an isomorphism (using the finite version of the Stable Elements Theorem)?
Some results for compact Lie groups

How can we prove that $\rho$ is an isomorphism (using the finite version of the Stable Elements Theorem)?

A Theorem by Friedlander and Mislin

Let $l \neq p$ be a prime. Then, there exist an algebraic group $G(\overline{F}_l)$ and a map $\varphi: BG(\overline{F}_l) \rightarrow BG$ such that

$$H^\ast(BG; \mathbb{F}_p) \xrightarrow{\varphi^*} H^\ast(BG(\overline{F}_l); \mathbb{F}_p)$$

is an isomorphism.
How can we prove that $\rho$ is an isomorphism (using the finite version of the Stable Elements Theorem)?

A Theorem by Friedlander and Mislin

Let $l \neq p$ be a prime. Then, there exist an algebraic group $G(\overline{F}_l)$ and a map $\varphi: BG(\overline{F}_l) \to BG$ such that

$$H^*(BG; \mathbb{F}_p) \xrightarrow{\varphi^*} H^*(BG(\overline{F}_l); \mathbb{F}_p)$$

is an isomorphism.

We can choose $G(\overline{F}_l)$ such that $S \in \text{Syl}_p(G(\overline{F}_l))$, and $\varphi|_{BS} = \text{Id}$. 
How can we prove that $\rho$ is an isomorphism (using the finite version of the Stable Elements Theorem)?

A Theorem by Friedlander and Mislin

Let $l \neq p$ be a prime. Then, there exist an algebraic group $G(\overline{F}_l)$ and a map $\varphi: BG(\overline{F}_l) \to BG$ such that

$$H^*(BG; \mathbb{F}_p) \xrightarrow{\varphi^*} H^*(BG(\overline{F}_l); \mathbb{F}_p)$$

is an isomorphism.

We can choose $G(\overline{F}_l)$ such that $S \in \text{Syl}_p(G(\overline{F}_l))$, and $\varphi|_{BS} = \text{Id}$.

Nontrivial Fact

$$\mathcal{F}_S(G) = \mathcal{F}_S(G(\overline{F}_l)).$$
For each $i \geq 1$, 

$$
G_i = G(F_i) \leq G(F_l).
$$

Then $G(F_l) = \bigcup_{i \geq 1} G_i$. 

Choose $S_i \in \text{Syl}_p(G_i)$ such that $S = \bigcup_{i \geq 1} S_i$. 

Form the category $FS_i(G_i)$. Then, 

$$
F_0 = \bigcup_{i \geq 0} FS_i(G_i) \subseteq FS(G). 
$$
For each $i \geq 1$,

Set $G_i = G(F/i) \leq G(F)$. Then $G(F) = \bigcup_{i \geq 1} G_i$.  

**Stable Elements Theorem for compact Lie groups**

$$H^*(BG; F_p) \sim = H^*(BG(G(F)); F_p) \sim = \lim_{\leftarrow} H^*(BG_i; F_p) \sim = \lim_{\leftarrow} (\lim_{\leftarrow} F_{Si}(G_i)) H^*(-; F_p) \sim = \lim_{\leftarrow} F_0 H^*(-; F_p) \sim = \lim_{\leftarrow} F_{S}(G) H^*(-; F_p) \leq H^*(BS; F_p).$$
For each $i \geq 1$,

1. Set $G_i = G(F_i) \leq G(F)$. Then $G(F) = \bigcup_{i \geq 1} G_i$.
2. Choose $S_i \in Syl_p(G_i)$ such that $S = \bigcup_{i \geq 1} S_i$. 
For each $i \geq 1$,

1. Set $G_i = G(\overline{F}_i) \leq G(\overline{F})$. Then $G(\overline{F}) = \bigcup_{i \geq 1} G_i$.
2. Choose $S_i \in S_{\text{yl}}_p(G_i)$ such that $S = \bigcup_{i \geq 1} S_i$.
3. Form the category $\mathcal{F}_{S_i}(G_i)$. Then, $\mathcal{F}^0 \overset{\text{def}}{=} \bigcup_{i \geq 0} \mathcal{F}_{S_i}(G_i) \subseteq \mathcal{F}_S(G)$.
For each $i \geq 1$,

- Set $G_i = G(F_{li}) \leq G(F_l)$. Then $G(F_l) = \bigcup_{i \geq 1} G_i$.
- Choose $S_i \in Syl_p(G_i)$ such that $S = \bigcup_{i \geq 1} S_i$.
- Form the category $\mathcal{F}_{S_i}(G_i)$. Then, $\mathcal{F}^0 \overset{\text{def}}{=} \bigcup_{i \geq 0} \mathcal{F}_{S_i}(G_i) \subseteq \mathcal{F}_S(G)$.

### Stable Elements Theorem for compact Lie groups

\[
H^*(BG; \mathbb{F}_p) \cong H^*(BG(G(F_l)); \mathbb{F}_p) \cong \lim_{\rightarrow i} H^*(BG_i; \mathbb{F}_p) \cong \]

\[\text{lim}_{\leftarrow i} H^*(BG^i; \mathbb{F}_p) \cong \]
For each $i \geq 1$,

1. Set $G_i = G(F_{\ell i}) \leq G(\overline{F}_{\ell})$. Then $G(\overline{F}_{\ell}) = \bigcup_{i \geq 1} G_i$.
2. Choose $S_i \in \text{Syl}_p(G_i)$ such that $S = \bigcup_{i \geq 1} S_i$.
3. Form the category $\mathcal{F}_{S_i}(G_i)$. Then, $\mathcal{F}_0 \overset{\text{def}}{=} \bigcup_{i \geq 0} \mathcal{F}_{S_i}(G_i) \subseteq \mathcal{F}_S(G)$.

**Stable Elements Theorem for compact Lie groups**

$$H^*(BG; \mathbb{F}_p) \cong H^*(BG(G(\overline{F}_{\ell})); \mathbb{F}_p) \cong \lim_{\leftarrow i} H^*(BG_i; \mathbb{F}_p) \cong$$

$$\cong \lim_{\leftarrow i} \left( \lim_{\leftarrow i} H^*(-; \mathbb{F}_p) \right) \cong$$
For each $i \geq 1$,

1. Set $G_i = G(F_i) \leq G(F)$. Then $G(F) = \bigcup_{i \geq 1} G_i$.
2. Choose $S_i \in \text{Syl}_p(G_i)$ such that $S = \bigcup_{i \geq 1} S_i$.
3. Form the category $\mathcal{F}_{S_i}(G_i)$. Then, $\mathcal{F}^0 \overset{\text{def}}{=} \bigcup_{i \geq 0} \mathcal{F}_{S_i}(G_i) \subseteq \mathcal{F}_S(G)$.

### Stable Elements Theorem for compact Lie groups

\[
H^*(BG; \mathbb{F}_p) \cong H^*(BG(G(F)); \mathbb{F}_p) \cong \lim_{\leftarrow i} H^*(BG_i; \mathbb{F}_p) \cong \\
\cong \lim_{\leftarrow i} \left( \lim_{\leftarrow i} H^*(-; \mathbb{F}_p) \right) \cong \\
\cong \lim_{\leftarrow \mathcal{F}^0} H^*(-; \mathbb{F}_p) \cong \lim_{\leftarrow \mathcal{F}_S(G)} H^*(-; \mathbb{F}_p) \leq H^*(BS; \mathbb{F}_p).
\]
There are other topological spaces that give rise to categories of the form $\mathcal{F}_S(G)$, but where no such group $G$ is involved.
Getting rid of the groups...

There are other topological spaces that give rise to categories of the form $\mathcal{F}_S(G)$, but where no such group $G$ is involved. The main example (besides compact Lie groups) is given by $p$-compact groups: $p$-complete loop spaces with finite mod $p$ cohomology.
There are other topological spaces that give rise to categories of the form $\mathcal{F}_S(G)$, but where no such group $G$ is involved. The main example (besides compact Lie groups) is given by $p$-compact groups: $p$-complete loop spaces with finite mod $p$ cohomology.

**$p$-local compact groups**

A $p$-local compact group is a triple $(S, \mathcal{F}, \mathcal{L})$, where
There are other topological spaces that give rise to categories of the form $\mathcal{F}_S(G)$, but where no such group $G$ is involved. The main example (besides compact Lie groups) is given by $p$-compact groups: $p$-complete loop spaces with finite mod $p$ cohomology.

### $p$-local compact groups

A $p$-local compact group is a triple $(S, \mathcal{F}, \mathcal{L})$, where

1. $S$ fits in a group extension $T \to S \to \pi$ where $T \cong (\mathbb{Z}/p^\infty)^{\times r}$ and $\pi$ is a finite group.
Getting rid of the groups...

There are other topological spaces that give rise to categories of the form $\mathcal{F}_S(G)$, but where no such group $G$ is involved. The main example (besides compact Lie groups) is given by $p$-compact groups: $p$-complete loop spaces with finite mod $p$ cohomology.

### $p$-local compact groups

A $p$-local compact group is a triple $(S, \mathcal{F}, \mathcal{L})$, where

1. $S$ fits in a group extension $T \rightarrow S \rightarrow \pi$ where $T \cong (\mathbb{Z}/p^\infty)^\times r$ and $\pi$ is a finite group.

2. $\mathcal{F}$ is a **saturated fusion system**: a category where $\text{Ob}(\mathcal{F}) = \{ P \leq S \}$, and whose morphisms satisfy a certain set of conditions.
There are other topological spaces that give rise to categories of the form $\mathcal{F}_S(G)$, but where no such group $G$ is involved. The main example (besides compact Lie groups) is given by $p$-compact groups: $p$-complete loop spaces with finite mod $p$ cohomology.

### $p$-local compact groups

A $p$-local compact group is a triple $(S, \mathcal{F}, \mathcal{L})$, where

1. $S$ fits in a group extension $T \to S \to \pi$ where $T \cong (\mathbb{Z}/p^\infty)^\times r$ and $\pi$ is a finite group.

2. $\mathcal{F}$ is a **saturated fusion system**: a category where $\text{Ob}(\mathcal{F}) = \{P \leq S\}$, and whose morphisms satisfy a certain set of conditions.

3. $\mathcal{L}$ is a **centric linking system**: a category where $\text{Ob}(\mathcal{L}) \subseteq \text{Ob}(\mathcal{L})$, and whose morphisms satisfy another set of conditions.
$\mathcal{F}$ satisfies the following conditions for all $P, Q \leq S$:

1. $\text{Hom}_S(P, Q) \subseteq \text{Mor}_\mathcal{F}(P, Q) \subseteq \text{Inj}(P, Q)$.
2. If $f \in \text{Hom}_\mathcal{F}(P, Q)$, then every homomorphism in the triangle is a morphism in $\mathcal{F}$.

In addition, there are three more conditions, inspired in the case $\mathcal{F}_S(G)$.
\( \mathcal{F} \) satisfies the following conditions for all \( P, Q \leq S \):

1. \( \text{Hom}_S(P, Q) \subseteq \text{Mor}_\mathcal{F}(P, Q) \subseteq \text{Inj}(P, Q) \).

2. If \( f \in \text{Hom}_\mathcal{F}(P, Q) \), then every homomorphism in the triangle is a morphism in \( \mathcal{F} \).

In addition, there are three more conditions, inspired in the case \( \mathcal{F}_S(G) \).

The category \( \mathcal{L} \) supplies the absence of an ambient group.
\( \mathcal{F} \) satisfies the following conditions for all \( P, Q \leq S \):

1. \( \text{Hom}_S(P, Q) \subseteq \text{Mor}_\mathcal{F}(P, Q) \subseteq \text{Inj}(P, Q) \).

2. if \( f \in \text{Hom}_\mathcal{F}(P, Q) \), then every homomorphism in the triangle is a morphism in \( \mathcal{F} \).

In addition, there are three more conditions, inspired in the case \( \mathcal{F}_S(G) \).

The category \( \mathcal{L} \) supplies the absence of an ambient group.

The **classifying space** of \( (S, \mathcal{F}, \mathcal{L}) \) is the space \( |\mathcal{L}|_p^{\wedge} \).
Examples

Every finite group determines a $p$-local finite group.
Examples

Every finite group determines a $p$-local finite group. Let $G$ be a finite group, and let $S \in \text{Syl}_p(G)$. We have already defined $\mathcal{F}_S(G)$ above, it remains to define $\mathcal{L}_S(G)$.
Every finite group determines a $p$-local finite group. Let $G$ be a finite group, and let $S \in \text{Syl}_p(G)$. We have already defined $\mathcal{F}_S(G)$ above, it remains to define $\mathcal{L}_S(G)$.

$$\text{Ob}(\mathcal{L}_S(G)) = \{ P \leq S \mid C_G(P) \cong Z(P) \times C_G'(P) \},$$
where $C_G'(P)$ has order prime to $p$. 

Every finite group determines a $p$-local finite group. Let $G$ be a finite group, and let $S \in \text{Syl}_p(G)$. We have already defined $\mathcal{F}_S(G)$ above, it remains to define $\mathcal{L}_S(G)$.

1. $\text{Ob}(\mathcal{L}_S(G)) = \{P \leq S \mid C_G(P) \cong Z(P) \times C'_G(P)\}$, where $C'_G(P)$ has order prime to $p$.

2. For all $P, Q \in \text{Ob}(\mathcal{L}_S(G))$, set

$$\text{Mor}_{\mathcal{L}_S(G)}(P, Q) = N_G(P, Q)/C'_G(P),$$

where $N_G(P, Q) = \{x \in G \mid xPx^{-1} \leq Q\}$. 

Theorem (Broto-Levi-Oliver)

Let $G$ be a compact Lie group, and let $S \in \text{Syl}_p(G)$. Then, there exists $(S, F_S(G), \mathcal{L}_S(G))$ such that $|\mathcal{L}_S(G)| \wedge p \cong (BG) \wedge p$.

Let $(X, BX, e)$ be a $p$-compact group, and let $S \in \text{Syl}_p(X)$. Then, there exists $(S, F_S(X), \mathcal{L}_S(X))$ such that $|\mathcal{L}_S(X)| \wedge p \cong BX \wedge p$. 

Alex González

Finite approximations of $p$-local compact groups
Examples

Every finite group determines a $p$-local finite group. Let $G$ be a finite group, and let $S \in \text{Syl}_p(G)$. We have already defined $\mathcal{F}_S(G)$ above, it remains to define $\mathcal{L}_S(G)$.

1. $\text{Ob}(\mathcal{L}_S(G)) = \{P \leq S \mid C_G(P) \cong Z(P) \times C'_G(P)\}$, where $C'_G(P)$ has order prime to $p$.

2. For all $P, Q \in \text{Ob}(\mathcal{L}_S(G))$, set

$$\text{Mor}_{\mathcal{L}_S(G)}(P, Q) = N_G(P, Q)/C'_G(P),$$

where $N_G(P, Q) = \{x \in G \mid xPx^{-1} \leq Q\}$.

Theorem (Broto-Levi-Oliver)

1. Let $G$ be a compact Lie group, and let $S \in \text{Syl}_p(G)$. Then, there exists $(S, \mathcal{F}_S(G), \mathcal{L}_S(G))$ such that $|\mathcal{L}_S(G)|^\wedge_p \simeq (BG)^\wedge_p$.

2. Let $(X, BX, e)$ be a $p$-compact group, and let $S \in \text{Syl}_p(X)$. Then, there exists $(S, \mathcal{F}_S(X), \mathcal{L}_S(X))$ such that $|\mathcal{L}_S(X)|^\wedge_p \simeq BX$. 
Some facts about $p$-local compact groups

Stable Elements Theorem for $p$-local finite groups (Broto-Levi-Oliver)

Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local finite group. Then,

$$H^*(|\mathcal{L}|_p; \mathbb{F}_p) \cong \lim_{\mathcal{F}} H^*(-; \mathbb{F}_p).$$
Some facts about $p$-local compact groups

<table>
<thead>
<tr>
<th>Stable Elements Theorem for $p$-local finite groups (Broto-Levi-Oliver)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local finite group. Then,</td>
</tr>
<tr>
<td>[ H^*(</td>
</tr>
</tbody>
</table>

Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local compact group.
Some facts about $p$-local compact groups

Stable Elements Theorem for $p$-local finite groups (Broto-Levi-Oliver)

Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local finite group. Then,

$$H^* (|\mathcal{L}|_p^\wedge; \mathbb{F}_p) \cong \lim_{\mathcal{F}} H^*(-; \mathbb{F}_p).$$

Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local compact group.

- If $(S, \mathcal{F}, \mathcal{L})$ comes from a compact Lie group or a $p$-compact group, then the Stable Elements Theorem holds.
Stable Elements Theorem for $p$-local finite groups (Broto-Levi-Oliver)

Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local finite group. Then,

$$H^* (| \mathcal{L} |^\wedge_p; \mathbb{F}_p) \cong \lim_{\substack{\leftarrow \mathcal{F}}} H^*(-; \mathbb{F}_p).$$

Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local compact group.

1. If $(S, \mathcal{F}, \mathcal{L})$ comes from a compact Lie group or a $p$-compact group, then the Stable Elements Theorem holds.

2. There are $p$-local compact groups which do not correspond to compact Lie groups or $p$-compact groups.
Some facts about $p$-local compact groups

Stable Elements Theorem for $p$-local finite groups (Broto-Levi-Oliver)

Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local finite group. Then,

$$H^*(|\mathcal{L}|_p^\wedge; \mathbb{F}_p) \cong \lim_{\leftarrow \mathcal{F}} H^*(-; \mathbb{F}_p).$$

Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local compact group.

1. If $(S, \mathcal{F}, \mathcal{L})$ comes from a compact Lie group or a $p$-compact group, then the Stable Elements Theorem holds.

2. There are $p$-local compact groups which do not correspond to compact Lie groups or $p$-compact groups.

Question:

Is there a general Stable Elements Theorem for $p$-local compact groups?
Some obstacles:

1. The proof for $p$-local finite groups uses certain bisets which are not available in the compact case.
2. The proof for compact Lie groups or $p$-compact groups uses transfer arguments which are not available for $p$-local compact groups. We need a different strategy.

Unstable Adams operations for $p$-local compact groups

Let $\zeta \in (\mathbb{Z} \wedge p) \times$. An unstable Adams operation of degree $\zeta$ for $(S, F, L)$ is a self-equivalence $\Psi: L \to L$ such that:

1. $\Psi$ preserves the structure of $L$;
2. $\Psi$ induces an automorphism of $S$, $\Psi: S \to S$; and
3. $\Psi(t) = t^\zeta$ for all $t \in T$. 

Alex González

Finite approximations of $p$-local compact groups
Unstable Adams operations

Some obstacles:

- The proof for $p$-local finite groups uses certain bisets which are not available in the compact case.

Unstable Adams operations for $p$-local compact groups

Let $\zeta \in (\mathbb{Z} \wedge p) \times$. An unstable Adams operation of degree $\zeta$ for $(S, F, L)$ is a self-equivalence $\Psi : L \to L$ such that

1. $\Psi$ preserves the structure of $L$;
2. $\Psi$ induces an automorphism of $S$, $\Psi : S \to S$; and
3. $\Psi(t) = t^\zeta$ for all $t \in T$.
Some obstacles:

1. The proof for $p$-local finite groups uses certain bisets which are not available in the compact case.
2. The proof for compact Lie groups or $p$-compact groups uses transfer arguments which are not available for $p$-local compact groups.
Some obstacles:

1. The proof for $p$-local finite groups uses certain bisets which are not available in the compact case.

2. The proof for compact Lie groups or $p$-compact groups uses transfer arguments which are not available for $p$-local compact groups.

We need a different strategy.
Unstable Adams operations

Some obstacles:

1. The proof for $p$-local finite groups uses certain bisets which are not available in the compact case.
2. The proof for compact Lie groups or $p$-compact groups uses transfer arguments which are not available for $p$-local compact groups.

We need a different strategy.

Unstable Adams operations for $p$-local compact groups

Let $\zeta \in (\mathbb{Z}_p^\wedge)^\times$. An unstable Adams operation of degree $\zeta$ for $(S, \mathcal{F}, \mathcal{L})$ is a self-equivalence $\Psi : \mathcal{L} \to \mathcal{L}$ such that

1. preserves the structure of $\mathcal{L}$;
2. $\Psi$ induces an automorphism of $S$, $\Psi : S \to S$; and
3. $\Psi(t) = t^\zeta$ for all $t \in T$. 

Alex González
Finite approximations of $p$-local compact groups

11/13
We can use unstable Adams operations to produce approximations of $p$-local compact groups by $p$-local finite groups.
We can use unstable Adams operations to produce approximations of $p$-local compact groups by $p$-local finite groups. Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local compact group, and let $\Psi$ be an unstable Adams operation.
We can use unstable Adams operations to produce approximations of $p$-local compact groups by $p$-local finite groups. Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local compact group, and let $\Psi$ be an unstable Adams operation. Set

\[ \Psi_0 = \Psi, \text{ and } \Psi_{i+1} = \Psi_i^p \text{ for all } i \geq 0. \]
We can use unstable Adams operations to produce approximations of $p$-local compact groups by $p$-local finite groups. Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local compact group, and let $\Psi$ be an unstable Adams operation. Set

1. $\Psi_0 = \Psi$, and $\Psi_{i+1} = \Psi_i^p$ for all $i \geq 0$.
2. $S_i = \{x \in S \mid \Psi_i(x) = x\}.$
We can use unstable Adams operations to produce approximations of $p$-local compact groups by $p$-local finite groups. Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local compact group, and let $\Psi$ be an unstable Adams operation. Set

1. $\Psi_0 = \Psi$, and $\Psi_{i+1} = \Psi_p^i$ for all $i \geq 0$.
2. $S_i = \{x \in S \mid \Psi_i(x) = x\}$.
3. $\mathcal{L}_i \subseteq \mathcal{L}$, the subcategory with $\text{Ob}(\mathcal{L}_i) = \{P \in \text{Ob}(\mathcal{L}) \mid P \leq S_i\}$ and $\text{Mor}_{\mathcal{L}_i}(P, Q) = \{\phi \in \text{Mor}_\mathcal{L}(P, Q) \mid \Psi_i(\phi) = \phi\}$. 
Finite approximations of $p$-local compact groups

We can use unstable Adams operations to produce approximations of $p$-local compact groups by $p$-local finite groups. Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local compact group, and let $\Psi$ be an unstable Adams operation. Set

1. $\Psi_0 = \Psi$, and $\Psi_{i+1} = \Psi^p_i$ for all $i \geq 0$.
2. $S_i = \{ x \in S \mid \Psi_i(x) = x \}$.
3. $\mathcal{L}_i \subseteq \mathcal{L}$, the subcategory with $\text{Ob}(\mathcal{L}_i) = \{ P \in \text{Ob}(\mathcal{L}) \mid P \leq S_i \}$ and $\text{Mor}_{\mathcal{L}_i}(P, Q) = \{ \phi \in \text{Mor}_{\mathcal{L}}(P, Q) \mid \Psi_i(\phi) = \phi \}$.
4. $\mathcal{F}_i \subseteq \mathcal{F}$, the fusion system over $S_i$ generated by $\mathcal{L}_i$. 
**Finite Approximation Theorem (G.)**

1. There exists $M \in \mathbb{N}$ such that $(S_i, F_i, L_i)$ is a $p$-local finite group for all $i \geq M$.
2. $|L|_p^\wedge \simeq (\text{hocolim} |L_i|_p^\wedge)_p$. 

---

Finite approximations of $p$-local compact groups
Finite approximations of $p$-local compact groups

**Finite Approximation Theorem (G.)**

1. There exists $M \in \mathbb{N}$ such that $(S_i, \mathcal{F}_i, \mathcal{L}_i)$ is a $p$-local finite group for all $i \geq M$.
2. $|\mathcal{L}|_p^\wedge \simeq (\hocolim |\mathcal{L}_i|_p^\wedge)_p$.

**Stable Elements Theorem for $p$-local compact groups**

$$H^*(|\mathcal{L}|_p^\wedge; \mathbb{F}_p) \simeq \lim_{\leftarrow i} H^*(|\mathcal{L}_i|_p^\wedge; \mathbb{F}_p) \simeq \lim_{\leftarrow i} (\lim_{\mathcal{F}_i} H^*(-; \mathbb{F}_p)) \simeq$$

$$\simeq \lim_{\mathcal{F}^0} H^*(-; \mathbb{F}_p) \simeq \lim_{\mathcal{F}} H^*(-; \mathbb{F}_p) \leq H^*(BS; \mathbb{F}_p).$$
The homotopy type of mapping spaces is difficult to study in general.
The homotopy type of mapping spaces is difficult to study in general. There is a particular situation where mapping spaces are reasonably easy to describe: mapping spaces between classifying spaces of finite groups.
The homotopy type of mapping spaces is difficult to study in general. There is a particular situation where mapping spaces are reasonably easy to describe: mapping spaces between classifying spaces of finite groups.

**Theorem**

Let \( f : H \to G \) be a homomorphism between discrete groups. Then

\[
\text{Map}(BH, BG)_{Bf} \simeq BC_G(f(H)).
\]
The homotopy type of mapping spaces is difficult to study in general. There is a particular situation where mapping spaces are reasonably easy to describe: mapping spaces between classifying spaces of finite groups.

**Theorem**

Let $f : H \to G$ be a homomorphism between discrete groups. Then

$$\text{Map}(BH, BG)_Bf \simeq BC_G(f(H)).$$

**Question**

Does this Theorem extend (in some sense) to $p$-local compact groups?
Let \((S, \mathcal{F}, \mathcal{L})\) be a \(p\)-local compact group. In general, there is no notion of centralizer of a \(p\)-local compact subgroup.
Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local compact group. In general, there is no notion of centralizer of a $p$-local compact subgroup. If $P \leq S$, then there is a well-defined centralizer of $P$, which is again a $p$-local compact group: $(C_S(P), C_{\mathcal{F}}(P), C_{\mathcal{L}}(P))$. 
Let \((S, \mathcal{F}, \mathcal{L})\) be a \(p\)-local compact group. In general, there is no notion of centralizer of a \(p\)-local compact subgroup. If \(P \leq S\), then there is a well-defined centralizer of \(P\), which is again a \(p\)-local compact group: \((C_S(P), C_\mathcal{F}(P), C_\mathcal{L}(P))\).

**Mapping Spaces and Centralizers Theorem (B.-L.-O., G.)**

Let \(P\) be a locally finite \(p\)-group satisfying the descending chain condition, and let \(\gamma: BP \to |\mathcal{L}|_p^\wedge\) be a map. Then,

\[
\text{Map}(BP, |\mathcal{L}|_p^\wedge)_\gamma \simeq |C_\mathcal{L}()\gamma(P))|_p^\wedge.
\]