1. Introduction

When we consider properties of a “reasonable” function, probably the first thing that comes to mind is that it exhibits **continuity**: the behavior of the function at a certain point is similar to the behavior of the function in a small neighborhood of the point. What’s more, the composition of two continuous functions is also continuous.

Usually, when we think of a continuous functions, the first examples that come to mind are maps $f : \mathbb{R} \to \mathbb{R}$:

- the identity function, $f(x) = x$ for all $x \in \mathbb{R}$;
- a constant function $f(x) = k$;
- polynomial functions, for instance $f(x) = x^n$, for some $n \in \mathbb{N}$;
- the exponential function $g(x) = e^x$;
- trigonometric functions, for instance $h(x) = \cos(x)$.

The set of real numbers $\mathbb{R}$ is a natural choice of domain to begin to study more general properties of continuous functions. After all, we are familiar with many of the properties of the real line, and it is (relatively) easy to draw the graphs of functions $\mathbb{R} \to \mathbb{R}$.

So, let’s review the definition of continuity for a function $f : \mathbb{R} \to \mathbb{R}$: the function $f$ is continuous at the point $a \in \mathbb{R}$ if

$$\lim_{x \to a} f(x) = f(a)$$

(in particular we are assuming that this limit exists!). Another way of putting the above definition is the following: for each $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \text{ for all } x \in \mathbb{R} \text{ such that } |x - a| < \delta.$$

The function $f$ is then said to be continuous on all $\mathbb{R}$ if it is continuous for all $a \in \mathbb{R}$. So, basically, the definition of continuity depends only on the notion of the distance between two points, codified in the expressions $|x - a|$, $|f(x) - f(a)|$.

Now consider functions $f : \mathbb{R}^n \to \mathbb{R}^m$. In this case, as you know, there is also a well-defined notion of continuity, which again depends only on the metrics that we have previously defined in $\mathbb{R}^n$ and $\mathbb{R}^m$. It is not a big step to say that the same applies for functions $\mathbb{C} \to \mathbb{C}$.

As it turns out, all these situations, and many more, fit together under the name of **metric spaces**: sets (like $\mathbb{R}$, $\mathbb{N}$, $\mathbb{R}^n$, etc) on which we can measure the distance between two points. The first goal of this course is then to define metric spaces and continuous functions between metric spaces.
A note of waning! The same set can be given different ways of measuring distances. Strange as it may seem, the set $\mathbb{R}^2$ (the plane) is one of these sets. We will see different metrics for $\mathbb{R}^2$ pretty soon.

You may have noticed that metrics also have something to do with the notion of convergence of sequences. In fact, whenever a space has a metric, we can talk about sequences and investigate their convergence. Let’s see an interesting example of this: fix an interval $[a, b]$, and consider the set $X$ of all differentiable functions $f : [a, b] \to \mathbb{R}$. We can define a metric (whatever this means) on $X$ as follows: given $f, g \in X$,

$$d(f, g) = \max_{x \in [a, b]} \{|f(x) - g(x)|\}.$$ 

With this metric on the function space (i.e. notion of distance between functions), we can define Taylor approximation in terms of converging sequences of functions on $X$.

Returning to continuous functions on one variable, there are a few more important properties of them which only depend on the distance between points. For example.

- **Intermediate Value Theorem** (a.k.a. Bolzano’s Theorem): if $f : [a, b] \to \mathbb{R}$ is continuous and $y \in \mathbb{R}$ is a number between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that $f(c) = y$.

- If $f : [a, b] \to \mathbb{R}$ is continuous, then there exists some $K \in \mathbb{R}$ such that $|f(x)| \leq K$ for all $x \in [a, b]$.

In the above statements, the domain of $f$ is the interval $[a, b]$, which is closed and bounded as a subset of $\mathbb{R}$, and this is all we need to prove this results. As you probably know already, the above results have their own versions for continuous functions $f : \mathbb{R}^n \to \mathbb{R}^m$, since there is well stablished notions of closed and bounded subsets in $\mathbb{R}^n$, for all $n$.

The second main goal of this course is to generalize the notions of closed and bounded, so that we can generalize these results for continuous maps between metric spaces.

And what about topology? What is topology anyway? It turns out that we don’t even need a metric to talk about continuity. In order to consider continuous functions between any spaces, all we must do is make clear from the beginning what the open and closed subsets of the spaces are. Topology gives us the rules for doing this. This is an abstract, powerful generalization of metric spaces, so be careful!
Let \( X \) be a set. Roughly speaking, a metric on the set \( X \) is just a rule to measure the distance between any two elements of \( X \).

**Definition 2.1.** A metric on the set \( X \) is a function \( d \colon X \times X \to [0, \infty) \) such that the following conditions are satisfied for all \( x, y, z \in X \):

- (M1) Positive property: \( d(x, y) = 0 \) if and only if \( x = y \);
- (M2) Symmetry property: \( d(x, y) = d(y, x) \); and
- (M3) Triangle inequality: \( d(x, y) \leq d(x, z) + d(z, y) \).

Let’s see some examples of metric spaces.

**Example 2.2.** The set \( X = \mathbb{R} \) with \( d(x, y) = |x - y| \), the absolute value of the difference \( x - y \), for each \( x, y \in \mathbb{R} \). Properties (M1) and (M2) are obvious, and

\[
d(x, y) = |x - y| = |(x - z) + (z - y)| \leq |x - z| + |z - y| = d(x, z) + d(z, y).
\]

Let’s see now some examples of metrics on the set \( X = \mathbb{R}^2 \).

**Example 2.3.** The set \( X = \mathbb{R}^2 \) with

\[
d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}
\]

for each \( (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2 \). Again, properties (M1) and (M2) are easy to check. Property (M3), the triangle inequality, gets its name from this example. Write \( x = (x_1, x_2), y = (y_1, y_2) \) and \( z = (z_1, z_2) \). Then,

Can you think of other examples of metrics on \( \mathbb{R}^2 \)? If we compare the metrics in Examples 2.2 and 2.3, we can see that the metric on \( \mathbb{R}^2 \) is some sort of “extension” of the metric on \( \mathbb{R}^1 \) to a higher dimension. But we can try other ways of extending it.

**Example 2.4.** The function \( d, d' \colon \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty) \) defined by

\[
d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|,
\]

\[
d'((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}
\]
are also metrics on $\mathbb{R}^2$ (the details will be checked in Examples 2.5 and 2.8 below).

We have just see three different metrics on $\mathbb{R}^2$. But why stopping at $\mathbb{R}^2$? We can extend these metrics to $\mathbb{R}^n$ for all $n \geq 1$.

**Example 2.5.** Let $X = \mathbb{R}^n$ and let $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ be defined by

$$d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \sum_{i=1}^{n} |x_i - y_i|.$$ 

Then, $d$ is a metric on $\mathbb{R}^n$ (sometimes known by the name of *Manhattan metric*). Let’s check the details: conditions (M1) and (M2) follow easily. As for the triangle inequality, we have

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sum_{i=1}^{n} |x_i - y_i| =$$

$$= \sum_{i=1}^{n} |(x_i - z_i) + (z_i - y_i)| \leq$$

$$\leq \sum_{i=1}^{n} |x_i - z_i| + |z_i - y_i| =$$

$$= d((x_1, \ldots, x_n), (z_1, \ldots, z_n)) + d((z_1, \ldots, z_n), (y_1, \ldots, y_n)).$$

**Example 2.6.** The *Euclidean metric* on $\mathbb{R}^n$ is defined by the formula

$$d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

for each $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. As usual (M1) and (M2) are easy to check, but (M3) is not trivial at all.

Indeed, let $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n)$ and $z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$: Let also $r_i = x_i - z_i$ and $s_i = z_i - y_i$, for $i = 1, \ldots, n$. We have to prove that

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (r_i + s_i)^2} \leq \sqrt{\sum_{i=1}^{n} r_i^2} + \sqrt{\sum_{i=1}^{n} s_i^2} = d(x, z) + d(z, y) \tag{1}$$

Notice that both sides of the inequality are positive. Thus, by squaring the above, it is equivalent to prove that

$$\sum_{i=1}^{n} (r_i + s_i)^2 \leq \sum_{i=1}^{n} r_i^2 + \sum_{i=1}^{n} s_i^2 + 2 \sqrt{\sum_{i=1}^{n} r_i^2 \sum_{i=1}^{n} s_i^2}. \tag{2}$$
The left part of the inequality expands to
\[ \sum_{i=1}^{n} (r_i + s_i)^2 = \sum_{i=1}^{n} r_i^2 + \sum_{i=1}^{n} s_i^2 + 2 \sum_{i=1}^{n} r_is_i. \]
Replacing this in (2) and simplifying, we deduce that (1) holds if and only if
\[ \left( \sum_{i=1}^{n} r_is_i \right)^2 \leq \left( \sum_{i=1}^{n} r_i^2 \right) \left( \sum_{i=1}^{n} s_i^2 \right). \]
The last inequality is the Cauchy-Schwartz inequality, which we prove below as Lemma 2.7.

**Lemma 2.7 (Cauchy-Schwartz inequality).** For each \((a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbb{R}^n\),
\[ \left( \sum_{i=1}^{n} a_ib_i \right)^2 \leq \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right). \]

**Proof.** Consider the expression \(\sum_{i=1}^{n} \sum_{j=1}^{n} (a_ib_j - a_jb_i)^2\). By expanding the brackets,
\[ \sum_{i=1}^{n} \sum_{j=1}^{n} (a_ib_j - a_jb_i)^2 = \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{j=1}^{n} b_j^2 \right) + \left( \sum_{j=1}^{n} a_j^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right) - 2 \left( \sum_{i=1}^{n} a_ib_i \right) \left( \sum_{j=1}^{n} a_jb_j \right). \]
Now, collect the terms (and reindex the sums) to get
\[ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (a_ib_j - a_jb_i)^2 = \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right) - \left( \sum_{i=1}^{n} a_ib_i \right)^2. \]
Since the left part of the equality is positive, this proves the statement. \(\square\)

**Example 2.8.** Finally, the **box metric** on \(\mathbb{R}^n\) is defined by
\[ d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max\{|x_i - y_i| \mid i = 1, \ldots, n\} \]
Properties (M1) and (M2) are easily seen to hold. Let's check property (M3). Let \(x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)\) and \(z = (z_1, \ldots, z_n) \in \mathbb{R}^n\). Then, for each \(i = 1, \ldots, n\),
\[ |x_i - y_i| = |(x_i - z_i) + (z_i - y_i)| \leq |x_i - z_i| + |z_i - y_i| \leq d(x, z) + d(z, y). \]
Thus, the triangle inequality holds. As an exercise, consider the set \(\mathbb{R}^2\) with this metric. Fix then a point \(a \in \mathbb{R}^2\) and draw the set
\[ D_a = \{x \in \mathbb{R}^2 \mid d(a, x) \leq 1\}. \]
All these examples should serve as a warning of how flexible the notion of metric is: we should not be surprised to see that statements that go against all intuition are, in fact, true. The following is a good examples of this: it is a process to define a metric on every set. This procedure is not rather descriptive, but it is nonetheless important.
Example 2.9. Let $X$ be any set. The discrete metric on $X$ is defined by 

$$d(x, y) = \begin{cases} 
1, & \text{if } x \neq y \\
0, & \text{if } x = y 
\end{cases}$$

Properties (M1) and (M2) are obvious, so let’s check property (M3). Let $x, y, z \in X$. If $x = y$, then $d(x, y) = 0$, and there is nothing to check. Suppose then that $x \neq y$. Then, either $z = x \neq y$, or $z = y \neq x$, or $z \neq x, y$. Regardless the situation, we have then

$$1 = d(x, y) \quad \text{and} \quad 1 \leq d(x, z) + d(z, y) \leq 2,$$

and the triangle inequality holds.

One could say that this metric is rather bad: if we think of a metric as a rule to measure the distance between points $x, y$ in the set $X$, then in this case all points are far (if $x \neq y$) or really close (if $x = y$), but we cannot distinguish anything beyond this. However, this metric is not useless: it becomes a nice tool to check if geometric intuition really works and provides counterexamples.

Exercise 2.10. Let $X$ be a set and let $d: X \times X \to [0, \infty)$ be a function satisfying the following conditions

- (M1) $d(x, y) = 0$ if and only if $x = y$; and
- (M3) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Show that $d$ is a metric on $X$.

Example 2.11. Consider the set of integer numbers $\mathbb{Z}$, and let $p$ be a prime number. The $p$-valuation of $n \in \mathbb{Z}$ is the maximum non-negative integer $\nu_p(n)$ such that $p^{\nu_p(n)}$ divides $n$.

We can now define the $p$-adic metric on $\mathbb{Z}$ by

$$d(a, b) = \begin{cases} 
0, & \text{if } a = b; \\
\frac{1}{p^{\nu_p(a-b)}}, & \text{otherwise}. 
\end{cases}$$

Property (M1) holds by definition of $d$, and property (M2) holds by definition of the $p$-adic valuation: $\nu_p(a - b) = \nu_p(b - a)$. Let’s prove the triangle inequality. Let then $a, b, c \in \mathbb{Z}$. We can assume that these are all different numbers. In this case, property (M3) looks like

$$\frac{1}{p^{\nu_p(a-b)}} \leq \frac{1}{p^{\nu_p(a-c)}} + \frac{1}{p^{\nu_p(c-b)}},$$

and this is a consequence of the following property of the $p$-adic valuation:

- $\nu_p(a - b) \geq \min\{\nu_p(a - c), \nu_p(c - b)\}$.

Let’s prove the above property. Let then $k = \nu_p(a - b)$, $m = \nu_p(a - c)$, $n = \nu_p(c - b)$ and $l = \min\{\nu_p(a - c), \nu_p(c - b)\}$. By definition of the $p$-adic valuation, there exist $x, y, z \in \mathbb{Z}$ such that $p$ does not divide any of them, and such that

$$p^k x = a - b = a - c + c - b = p^m y + p^n z = p^l (p^{m-l} y + p^{n-l} z),$$
and now it is clear that \( k \geq l \).

**Exercise 2.12.** Consider \( \mathbb{Z} \) with the 2-adic metric. For which elements \( n \in \mathbb{Z} \) the distance between \( n \) and 1 is \( \frac{1}{2} \)?

**Example 2.13.** Let \( (X, d) \) be a metric space and let \( H \subseteq X \) be a non-empty subset. We can then restrict the metric \( d \) to the subset \( H \): \( d_H = d|_{H \times H} : H \times H \to [0, \infty) \). In other words,

\[
d_H(x, y) = d(x, y) \text{ for all } x, y \in H.
\]

It is clear that \( d_H \) is a metric on \( H \). We call \( d_H \) the **induced metric** on \( H \), and this makes \( (H, d_H) \) into a **metric subspace** of \( (X, d) \).

For example, for each \( n \geq 1 \), \( \mathbb{R}^n \sim R^n \times \{1\} \subseteq \mathbb{R}^{n+1} \). This way we can see the Euclidean metric on \( \mathbb{R}^n \) as the induced metric from the Euclidean metric on \( \mathbb{R}^{n+1} \).

2.1. **Normed real vector spaces.** An special and important example of metric spaces comes from real vector spaces with a norm. We have already seen an example of this: \( \mathbb{R} \) with the absolute value, but this example is way more general than that, and it deserves its own section.

**Definition 2.14.** Let \( V \) be a real vector space. A **norm** on \( V \) is a function \( \rho : V \to \mathbb{R} \) satisfying the following conditions for any \( u, v \in V \) and any \( a \in \mathbb{R} \):

1. \( \rho(u) = 0 \) if and only if \( u = 0 \);
2. \( \rho(au) = |a|\rho(u) \); and
3. (triangle inequality) \( \rho(u + v) \leq \rho(u) + \rho(v) \).

Combining the three conditions on \( \rho \), we deduce \( \rho(u) \geq 0 \) for all \( u \in \mathbb{R} \):

\[
0 = \rho(u - u) \leq \rho(u) + \rho(-u) = \rho(u) + \rho(u) = 2\rho(u).
\]

Given a normed vector space \( (V, \rho) \), we can now define the **norm metric**:

\[
\begin{align*}
\begin{array}{ccc}
V \times V & \xrightarrow{d} & [0, \infty) \\
(u, v) & \mapsto & \rho(u - v).
\end{array}
\end{align*}
\]

**Proposition 2.15.** The norm metric is indeed a metric on the set \( V \).

**Proof.** We have to check that the function \( d \) in (3) satisfies conditions (M1), (M2) and (M3) in Definition 2.1.

(M1) \( d(u, v) = 0 \) if and only if \( u = v \). Since \( d(u, v) = \rho(u - v) \), this property holds by property (i) of norms.

(M2) \( d(u, v) = d(v, u) \). This follows by property (ii) of norms:

\[
d(u, v) = \rho(u - v) = \rho((-1)(v - u)) = |-1|\rho(v - u) = d(v, u).
\]
(M3) \( d(u, v) \leq d(u, w) + d(w, v) \). This follows from the triangle inequality for norms:
\[
d(u, v) = \rho(u - v) = \rho((u - w) + (w - v)) \leq \rho(u - w) + \rho(w - v) = d(u, w) + d(w, v).
\]

□

Let’s see an application of this.

**Example 2.16.** The set of complex numbers \( \mathbb{C} \) is a real vector space of dimension 2. The modulus function, defined by
\[
\rho(a + bi) = \sqrt{a^2 + b^2}
\]
induces a metric on \( \mathbb{C} \). Obviously, this metric is the same as the Euclidean metric on \( \mathbb{R}^2 \cong \mathbb{C} \). In general, \( \mathbb{R}^n \) with the norm
\[
\rho(x_1, \ldots, x_n) = \sqrt{\sum_{i=1}^{n} x_i^2}
\]
induces the Euclidean metric on \( \mathbb{R}^n \).

**Example 2.17.** Let \( m \geq 2 \) be a fixed, natural number, and consider \( \mathbb{R}^n \) as a real vector space. Define then a map \( \rho : \mathbb{R}^n \to \mathbb{R} \) by
\[
\rho(x_1, \ldots, x_n) = m \sqrt[n]{\sum_{i=1}^{n} |x_i|^m}.
\]
Then \( \rho \) is a norm on \( \mathbb{R}^n \), and the induced norm metric \( d \) is the \( l_m \)-metric:
\[
d(x, y) = m \sqrt[n]{\sum_{i=1}^{n} |x_i - y_i|^m}.
\]

Do not get the wrong impression, not all metrics on a real vector space \( V \) come from a norm.

**Exercise 2.18.** Let \( V \) be a real vector space of dimension at least 1. Prove that there does not exist any norm on \( V \) inducing the discrete metric.

### 2.2. Product spaces.

Consider now the following situation: \( (X, d_X) \) and \( (Y, d_Y) \) are metric spaces, and \( X \times Y \) is the direct product set. It is natural now to ask if the metrics of \( X \) and \( Y \) can be put together to form a metric on \( X \times Y \), but the simplicity of this question hides a rather important problem: there is more than one way of combining \( d_X \) and \( d_Y \) to form a metric for \( X \times Y \), and none of these constructions is “more natural” than the others!

Define then maps \( d_1, d_2, d_\infty : (X \times Y) \times (X \times Y) \to [0, \infty) \) by the following formulas
\[
d_1((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)
\]
Proposition 2.19. Each of the maps \( d_1, d_2, d_\infty \) is a metric on \( X \times Y \).

The metric \( d_2 \) defined above is usually known as the Euclidean product metric, while \( d_\infty \) above is called the box product metric.

Proof. Properties (M1) and (M2) are easy to check in all three cases. Let’s check the triangle inequality for \( d_2 \), since the proof is similar for \( d_1 \) and \( d_\infty \). In this case, \( d_2 \) is the composition

\[
(X \times Y) \times (X \times Y) \xrightarrow{\cong} (X \times X) \times (Y \times Y) \xrightarrow{d_X \times d_Y} [0, \infty) \times [0, \infty) \xrightarrow{\text{incl}} \mathbb{R} \times \mathbb{R} \xrightarrow{\rho_{\text{Euc}}} [0, \infty)
\]

where \( \rho_{\text{Euc}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is the Euclidean norm on the real vector space \( \mathbb{R} \times \mathbb{R} \). It follows then

\[
d_2((x_1, y_1), (x_2, y_2)) = \rho_{\text{Euc}}(d_X(x_1, x_2), d_Y(y_1, y_2)) \leq
\]

\[
\leq \rho_{\text{Euc}}(d_X(x_1, x_3) + d_X(x_3, x_2), d_Y(y_1, y_3) + d_Y(y_3, y_2)) \leq
\]

\[
\leq \rho_{\text{Euc}}(d_X(x_1, x_3), d_Y(y_1, y_3)) + \rho_{\text{Euc}}(d_X(x_3, x_2), d_Y(y_3, y_2)) =
\]

\[
d_2((x_1, y_1), (x_3, y_3)) + d_2((x_3, y_3), (x_2, y_2))
\]

where the first inequality follows from the triangle inequality property of \( d_X \) and \( d_Y \), and the second inequality follows from the triangle inequality for norms.

Corollary 2.20. For every \((x_1, y_1), (x_2, y_2) \in X \times Y\),

\[
d_\infty((x_1, y_1), (x_2, y_2)) \leq d_2((x_1, y_1), (x_2, y_2)) \leq d_1((x_1, y_1), (x_2, y_2)).
\]

To understand why this poses a deep problem, consider the following situation. Let \( X = \mathbb{R} = Y \). In this case, the absolute value metric, the Euclidean metric, and the box metric are all the same. So, which metric should we choose for \( \mathbb{R}^2 = X \times Y \)? It is impossible to answer this question in this strict terms. As we will see in the next chapter, the answer is that the above three metrics, (4), (5) and (6), are equivalent under certain relation.

Remark 2.21. The same constructions and results apply to a finite collection of metric spaces \( \{(X_j, d_j)\} \) with the obvious modifications.
3. Open subsets

In the previous chapter we left an important problem unsolved: given metric spaces \((X, d_X)\) and \((Y, d_Y)\), is there a right choice of a metric for \(X \times Y\)? The answer to this question lies in the notion of open subsets of metric spaces: two metrics are equivalent if they define the same open subsets.

We will start by defining open subsets (with respect to a given metric). With this we can then define what the right metric on a product means. All this work to decide which metric we give to \(X \times Y\) will reveal a crucial fact: it is not so important the metric as it is the collection of open subsets that it defines! (in later chapters we will see how true this statement is). And so we will spend the second part of this chapter studying open subsets: we need to understand them before we can go further. In the last part of this chapter we will apply our recently acquired skills to study convergence of sequences in metric spaces.

Let’s then define what open subsets of a metric space are. Consider first \(\mathbb{R}^2\) with the Euclidean metric. What do you consider an open subset in \(\mathbb{R}^2\) with this metric? Intuitively, an open subset is a subset in which “one can get as close to the boundary as possible, but never reach it”. This is a fairly illustrative point of view, but it is not straightforward to formalize without getting tangled with technicalities. For example, how do we define the boundary of a subset?

Let’s analyze this situation from a different angle. The most characteristic example of open subset in \(\mathbb{R}^2\) (with Euclidean metric) is the so-called (open) ball of radius \(r\) around a point \(a \in \mathbb{R}^2\):

\[
B_{a,r} = \{ x \in \mathbb{R}^2 \mid d(a, x) < r \}
\]

There is then a rather easy way to formalize what an open subset of \(\mathbb{R}^2\) just in terms of balls of different radiiuses. Indeed, this idea of “getting closer and closer to the
boundary without reaching it” can be stated as follows. A subset $U \subseteq \mathbb{R}^2$ is open with respect to the Euclidean metric if, for each $y \in U$, there is some $\varepsilon \in (0, \infty)$ such that the open ball $B_{y,\varepsilon}$ is contained in $U$ (do you see any similarity with the notion of limit of a sequence?). It is clear now how to extend this definition to any metric space.

**Definition 3.1.** Let $(X, d)$ be a metric space. For each element $a \in X$ and each $\varepsilon \in (0, \infty) \subseteq \mathbb{R}$, the open ball with center $a$ and radius $\varepsilon$ is the subset $$B_{a,\varepsilon} = \{x \in X \mid d(a, x) < \varepsilon\} \subseteq X.$$  

A subset $U \subseteq X$ is open with respect to the metric $d$ if for each $y \in U$ there is some $\varepsilon(y) \in (0, \infty)$ such that $$B_{y,\varepsilon(y)} \subseteq U.$$  

A subset $U \subseteq X$ is closed with respect to the metric $d$ if the set $X \setminus U$ is open with respect to the metric $d$.

The notation $\varepsilon(y)$ above is there to stress the idea that the radius of the ball $B_{y,\varepsilon}$ depends on the element $y$. Note also that we have to specify with respect to which metric the subset $U$ is open in $X$. The following example illustrates the reason.

**Example 3.2.** Let $d_{\text{Euc}}$ and $d_{\text{dis}}$ be, respectively, the Euclidean and the discrete metrics on $\mathbb{R}^2$, and let $a \in \mathbb{R}^2$ be any point. Then, $U = \{a\} \subseteq \mathbb{R}^2$ is open with respect to $d_{\text{dis}}$ but not open with respect to $d_{\text{Euc}}$.

In the first exercise list we have already seen some examples of balls of radius 1 in $\mathbb{R}^2$ with respect to different metrics, so we should be aware by now that we cannot trust our senses on this.

### 3.1. Equivalent metrics.

The definition of open subset provides a comparison criteria for different metrics over the same set.

**Definition 3.3.** Let $X$ be a set, and let $d, d'$ be two metrics on $X$. We say that $d$ and $d'$ are topologically equivalent, $d \sim_{\text{top}} d'$, if, for every subset $U \subseteq X$,

$$U \text{ is open in } (X, d) \iff U \text{ is open in } (X, d').$$

In other words, two metrics are topologically equivalent if they define the same collections of open subsets on $X$. For example, the Euclidean metric and the discrete metric on $\mathbb{R}^2$ are not equivalent (see Example 3.2).

**Lemma 3.4.** The relation $\sim_{\text{top}}$ on the collection of all metrics of the set $X$ is an equivalence relation.

**Proof.** It is obvious. □
Let’s go back to the question of defining a metric on $X \times Y$, given metric spaces $(X, d_X)$ and $(Y, d_Y)$. Let then $d_1$, $d_2$ and $d_\infty$ be the metrics on $X \times Y$ defined in (4), (5) and (6). We will now prove that these three metrics are topologically equivalent. First we need a little result.

**Lemma 3.5.** Let $d_1$, $d_2$ and $d_\infty$ be the following metrics on $\mathbb{R}^2$:

- $d_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$;
- $d_2((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$;
- $d_\infty((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$.

Then, for each $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$,

$$\frac{1}{2} d_1((x_1, y_1), (x_2, y_2)) \leq \frac{1}{\sqrt{2}} d_2((x_1, y_1), (x_2, y_2)) \leq d_\infty((x_1, y_1), (x_2, y_2)).$$

**Proof.** By definition of the metric $d_2$,

$$d_2((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \leq$$

$$\leq \sqrt{\max\{|x_1 - x_2|^2, |y_1 - y_2|^2\} + \max\{|x_1 - x_2|^2, |y_1 - y_2|^2\}} =$$

$$= \sqrt{2 \max\{|x_1 - x_2|^2, |y_1 - y_2|^2\}} =$$

$$= \sqrt{2} d_\infty((x_1, y_1), (x_2, y_2)).$$

That proves the inequality between $d_2$ and $d_\infty$. Let’s now compare the metrics $d_1$ and $d_2$. Since everything is positive, proving the inequality between $d_1$ and $d_2$ in the statement is equivalent to proving the resulting inequality after squaring both sides. In other words, we have to prove

$$\frac{1}{4} (|x_1 - x_2| + |y_1 - y_2|)^2 \leq \frac{1}{2} ((x_1 - x_2)^2 + (y_1 - y_2)^2).$$

Let $a = |x_1 - x_2|$ and $b = |y_1 - y_2|$. Then, the above inequality is the same as

$$(a + b)^2 \leq 2(a^2 + b^2),$$

which is equivalent to $a^2 + b^2 \leq a^2 + b^2 + (a - b)^2$, and this last inequality is obviously true. Therefore, the inequality between $d_1$ and $d_2$ in the statement is true. □

**Theorem 3.6.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, and let $d_1$, $d_2$ and $d_\infty$ be the metrics on $X \times Y$ defined in (4), (5) and (6). Then, $d_1 \sim_{\text{top}} d_2 \sim_{\text{top}} d_\infty$.

**Proof.** For each $(x_1, y_1), (x_2, y_2) \in X \times Y$, let $u = (d_X(x_1, x_2), d_Y(y_1, y_2)) \in \mathbb{R}^2$. Then, we have

$$d_1((x_1, y_1), (x_2, y_2)) = d_{\mathbb{R}^2}^X(u, 0),$$

$$d_2((x_1, y_1), (x_2, y_2)) = d_{\mathbb{R}^2}^Y(u, 0),$$

$$d_\infty((x_1, y_1), (x_2, y_2)) = d_{\mathbb{R}^2}(u, 0).$$
and combining Corollary 2.20 and Lemma 3.5 we have then
\[
\frac{1}{2} \; d_1((x_1, y_1), (x_2, y_2)) \leq \frac{1}{\sqrt{2}} \; d_2((x_1, y_1), (x_2, y_2)) \leq d_\infty((x_1, y_1), (x_2, y_2)) \leq d_2((x_1, y_1), (x_2, y_2)) \leq d_1((x_1, y_1), (x_2, y_2)).
\]

Let now \( B_{a, \varepsilon}^{d_1}, B_{a, \varepsilon}^{d_2} \) and \( B_{a, \varepsilon}^{d_\infty} \) be the open balls of radius \( \varepsilon \) around \( a \in X \times Y \) with respect to the metrics \( d_1, d_2 \) and \( d_\infty \) respectively. The above inequalities imply that
\[
B_{a, \varepsilon}^{d_1} \subseteq B_{a, \varepsilon}^{d_2} \subseteq B_{a, \varepsilon}^{d_\infty} \subseteq B_{a, \sqrt{2} \varepsilon}^{d_2} \subseteq B_{a, 2\varepsilon}^{d_1},
\]
and the theorem follows from this. \( \square \)

Thus, even if \( d_1, d_2 \) and \( d_\infty \) are different metrics on \( X \times Y \), they are all equivalent. Thus it seems that the “right” metric for \( X \times Y \) is any metric in the equivalence class of \( d_1 \).

**Exercise 3.7.** Lemma 3.5 is stated only for \( \mathbb{R}^2 \) for simplicity, but there is a general statement for \( \mathbb{R}^n \). Namely, let \( d_1, d_2 \) and \( d_\infty \) be the corresponding metrics on \( \mathbb{R}^n \) (see Example 2.17 if you are not sure about how they are defined). Then, for each \( x, y \in \mathbb{R}^n \),
\[
\frac{1}{n} d_1(x, y) \leq \frac{1}{\sqrt{n}} d_2(x, y) \leq d_\infty(x, y).
\]
Can you prove these inequalities? They play the same role as Lemma 3.5 when proving Theorem 3.6 for a product of more than two metric spaces.

To finish the discussion about giving a metric to \( X \times Y \), we have to formalize what we understand by the “right metric”. Let’s start by defining an order relation in the collection of metrics on a given set.

**Definition 3.8.** Let \( X \) be a set, and let \( d, d' \) be metrics on \( X \). We say that \( d \) is a **weaker metric than** \( d' \) (and \( d' \) is an **stronger metric than** \( d \)) if every open subset of \( X \) with respect to \( d \) is also open with respect to \( d' \).

**Theorem 3.9.** Let \((X, d_X) \) and \((Y, d_Y) \) be metric spaces, and let \( d \) be a metric on \( X \times Y \) satisfying the following two conditions:

1. for each \( w \in Y \), the map \( i_w : X \to X \times Y \) defined by \( i_w(x) = (x, w) \) satisfies
   \[ d(i_w(x), i_w(x')) = d_X(x, x') \text{ for all } x, x' \in X \]
2. for each \( z \in X \), the map \( j_z : Y \to X \times Y \) defined by \( j_z(y) = (z, y) \) satisfies
   \[ d(j_z(y), j_z(y')) = d_Y(y, y') \text{ for all } y, y' \in Y. \]

Then the metric \( d \) is **weaker than** the metric \( d_1 \) (see (4)) defined by
\[
d_1((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).
\]
Roughly speaking, conditions (1) and (2) above mean that the metrics induced by \( d \) on \( X \) and \( Y \) are \( d_X \) and \( d_Y \) respectively. The theorem is then telling us what the expression “the right metric” means: given metrics \( d_X \) and \( d_Y \) for \( X \) and \( Y \) respectively, the metric \( d_1 \) is the strongest metric on \( X \times Y \) inducing \( d_X \) and \( d_Y \) on \( X \) and \( Y \) respectively.

**Proof.** Fix some \( u = (a, b) \in X \times Y \). Then, for every \( (x, y) \in X \times Y \) we can form the triangle below. The triangle inequality implies then the following.

\[
d(u, (x, y)) \leq d((a, b), (x, b)) + d((x, b), (x, y)) =
\]

\[
= d(i_b(a), i_b(x)) + d(j_x(b), j_x(y)) =
\]

\[
= d_X(a, x) + d_Y(b, y) = d_1(u, (x, y))
\]

Thus, for all \( u \in X \times Y \) and for all \( \varepsilon > 0 \), \( B_{u,\varepsilon}^{d_1} \subseteq B_{u,\varepsilon}^{d} \), and \( d_1 \) is stronger than \( d \). □

### 3.2. Properties of open subsets and a bit of set theory.

The discussion about giving a metric to a product of sets has pointed out the importance of open subsets with respect to a given metric. In fact, open subsets are even more important than we yet know, and thus it is the purpose of this section to learn some basic properties and practice with open subsets.

A little hint for this section: a little drawing always help to illustrate and understand a statement about sets. Although many of the examples that we have seen already contradict our intuition, a drawing can help see how to approach a possible proof.

Let’s start by describing the most important and basic properties of the collection of open subsets of a metric space.

**Theorem 3.10.** Let then \((X, d)\) be a metric space, and let \(\mathcal{O}\) be the collection of all open subsets of \(X\) with respect to \(d\). Then,

1. the total subset, \(X\), and the empty set, \(\emptyset\), are elements of \(\mathcal{O}\);
2. if \(\{U_\alpha\}_{\alpha \in \Gamma} \subseteq \mathcal{O}\) is any family (possibly infinite) of open subsets, then

\[
\bigcup_{\alpha \in \Gamma} U_\alpha \in \mathcal{O};
\]
3. if \(\{U_1, \ldots, U_n\} \subseteq \mathcal{O}\) is any finite family of open subsets, then

\[
\bigcap_{i=1}^n U_i \in \mathcal{O}.
\]
Proof. Property (1) is easy: since \( X \) contains all possible open balls, it is open. Also, the empty set \( \emptyset \) contains no points at all, and thus it also satisfies the condition to be an open subset.

Let now \( \mathcal{U} = \{ U_{\alpha} \}_{\alpha \in \Gamma} \subseteq \emptyset \) be a family (possibly infinite) of open subsets of \( X \), and let \( V = \bigcup_{\alpha} U_{\alpha} \subseteq X \). For each \( u \in V \), there is at least one \( U_{\alpha} \in \mathcal{U} \) which contains \( u \). Furthermore, since \( U_{\alpha} \) is open, there exists some \( \varepsilon > 0 \) such that 
\[
B_{d,u,\varepsilon} \subseteq U_{\alpha} \subseteq V.
\]
Hence \( V \) is open with respect to \( d \).

Finally, let \( \mathcal{U} = \{ U_{1}, \ldots, U_{n} \} \subseteq \emptyset \) be a finite family of open subsets. Let also \( V = \bigcap_{i=1}^{n} U_{i} \), and let \( u \in V \). Since \( u \in U_{i} \) for all \( i \), there exist \( \varepsilon_{1}, \ldots, \varepsilon_{n} > 0 \) such that 
\[
B_{d,u,\varepsilon_{i}} \subseteq U_{i}
\]
for all \( i \). Let then \( \varepsilon = \min\{ \varepsilon_{1}, \ldots, \varepsilon_{n} \} \). Then, \( B_{d,u,\varepsilon} \subseteq B_{d,u,\varepsilon_{i}} \) for all \( i \), and thus 
\[
B_{d,u,\varepsilon} \subseteq V
\]
and \( V \) is open with respect to \( d \). \( \square \)

Example 3.11. The following is an easy example of how property (iii) fails when we consider the intersection of an infinite family of open subsets. Let \( X = \mathbb{R} \) with the Euclidean metric, \( d(x, y) = |x - y| \), and let \( U_{i} = (-\frac{1}{i}, \frac{1}{i}) \) for all \( i \geq 1 \). Then,
\[
\bigcap_{i=1}^{\infty} U_{i} = \{ 0 \}.
\]
The properties describe in Theorem 3.10 can also be stated in terms of closed subsets.

Proposition 3.12. Let \((X, d)\) be a metric space. Then, the following holds:

1. \( X \) and \( \emptyset \) are closed subsets;
2. if \( \{ U_{1}, \ldots, U_{n} \} \subseteq \emptyset \) is any finite family of closed subsets, then the union \( \bigcup_{i=1}^{n} U_{i} \) is also a closed subset;
3. if \( \{ U_{\alpha} \}_{\alpha \in \Gamma} \subseteq \emptyset \) is any family (possibly infinite) of closed subsets, then the intersection \( \bigcap_{\alpha \in \Gamma} U_{\alpha} \) is also a closed subset.

Example 3.13. Let \( X \) be a set with the discrete metric. Then, for each \( x \in X \) the one-element subset \( \{ x \} \subseteq X \) is both open and closed. To check that it is open just notice that \( \{ x \} = B_{x,1} \) (open ball of radius 1). To check that it is closed, notice that the complement
\[
X \setminus \{ x \} = \bigcup_{y \in X \setminus \{ x \}} \{ y \} = \bigcup_{y \in X \setminus \{ x \}} B_{y,1}
\]
is a (possibly infinite) union of open subsets, and hence is open.
Example 3.14. Consider $\mathbb{R}^2$ with the Euclidean metric. Then, for each $u \in \mathbb{R}^2$, the subset $\{u\} \subseteq \mathbb{R}^2$ is closed but not open. Indeed, it is clear that $\{u\}$ is not open. To check that it is closed, consider the complement $U = \mathbb{R}^2 \setminus \{u\}$. We can describe $U$ as

$$U = \{v \in \mathbb{R}^2 \mid d(u, v) > 0\}.$$

Fix some $v \in U$ and set $\varepsilon = \frac{d(u, v)}{2} > 0$. Then, for each $w \in B_{v, \varepsilon}$ we have

$$d(u, w) \geq |d(u, v) - d(v, w)| = d(u, v) - d(v, w) > \frac{d(u, v)}{2} > 0,$$

and thus $w \in U$ (in the above list of (in)equalities we are using some result from the first exercise list).

Exercise 3.15. Let $(X, d)$ be a metric space and let $x \in X$ be any element. Prove that the subset $\{x\} \subseteq X$ is closed with respect to $d$ (in other words, regardless of the metric, a point is always a closed subset).

We have not given any particular name to the collection of closed subsets of a metric space: since the property of being closed is defined in terms of the property of being open, it is enough to understand the collection of open subsets $\emptyset$. However, this reasoning works also in the opposite direction: if we understand the collection of closed subsets then we understand the collection of open subsets, and in some particular cases it will be easier to work with closed subsets.

Let $(X, d)$ be a metric space. Which subsets of $X$ do you know to be either open and/or closed? From Theorem 3.10 and Proposition 3.12 we know that

- both $X$ and $\emptyset$ are open AND closed subsets.

Also, from Exercise 3.15 we know that, for each $x \in X$,

- $\{x\}$ is a closed subset and $X \setminus \{x\}$ is an open subset.

And what about open balls? after all, why calling them “open balls” if they turn out not to be open? that would make no sense! Well, indeed open balls are open subsets, but this requires a little proof.

Lemma 3.16. Let $(X, d)$ be a metric space. Then for each $x \in X$ and for each $\varepsilon > 0$,

1. $B_{x, \varepsilon} = \{y \in X \mid d(x, y) < \varepsilon\}$ is an open subset of $X$ with respect to $d$; and
2. $D_{x, \varepsilon} = \{y \in X \mid d(x, y) \leq \varepsilon\}$ is a closed subset of $X$ with respect to $d$.

Proof. To prove part (1), we have to show that, for each $y \in B_{x, \varepsilon}$ there exists some $\delta > 0$ such that $B_{y, \delta} \subseteq B_{x, \varepsilon}$. The following sketch already hints which is the right choice of $\delta$. 
Let then $\delta = \varepsilon - d(x, y) > 0$. Then, for each $z \in B_{y, \delta}$ the triangle inequality implies
\[
d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \delta = d(x, y) + \varepsilon - d(x, y) = \varepsilon
\]
and hence $z \in B_{x, \varepsilon}$. This proves that the subset $B_{x, \varepsilon}$ is open.

To prove (2), notice that $X \setminus D_{x, \varepsilon} = \{y \in X \mid d(x, y) > \varepsilon\}$, and we have to prove that this subset is open. Let then $y \in X \setminus D_{x, \varepsilon}$, as in the following picture.

Let $\delta = d(x, y) - \varepsilon > 0$, and suppose that for some $z \in B_{y, \delta}$, we have $d(x, z) \leq \varepsilon$. Then
\[
\varepsilon \geq d(x, z) \geq |d(x, y) - d(y, z)| = |
\varepsilon + \delta - d(y, z)| > \varepsilon + \delta - \delta = \varepsilon,
\]
where the second inequality (starting from the left) follows from Exercise 1 in List 1. But this is impossible, and thus for each $z \in B_{y, \delta}$ we have $d(x, z) > \varepsilon$. Thus $D_{x, \varepsilon}$ is closed.

A metric space $(X, d)$ may contain many subsets which are neither open nor closed, as we have seen already in the second exercise list. What should we do with such subsets? We can always relate them to open and closed subsets using the following definitions.
Definition 3.17. Let \((X, d)\) be a metric space, and let \(A \subseteq X\) be a subset.

- The **interior** of \(A\), \(\text{Int}(A)\), is the greatest open subset of \(X\) contained in \(A\).
- The **closure** of \(A\), \(\text{Cl}(A)\), is the smallest closed subset of \(X\) which contains \(A\).
- The **boundary** of \(A\), \(\partial A\), is the intersection \(\text{Cl}(A) \cap \text{Cl}(X \setminus A)\).

Thus, by definition, \(\text{Int}(A) \subseteq A \subseteq \text{Cl}(A)\) and \(\partial A = \partial (X \setminus A)\).

The following is an alternative characterization of the interior and the closure of a subset.

Lemma 3.18. Let \((X, d)\) be a metric space, and let \(A \subseteq X\). Then

1. The interior of \(A\) is the union of all open subsets of \(X\) contained in \(A\):
   \[
   \text{Int}(A) = \bigcup \{U \text{ open}, U \subseteq A\}.
   \]

2. The closure of \(A\) is the intersection of all closed subsets of \(X\) which contain \(A\):
   \[
   \text{Cl}(A) = \bigcap \{V \text{ closed}, A \subseteq V\}.
   \]

Proof. We do the proof for (1) and leave the proof of (2) as an exercise. Let first \(x \in \text{Int}(A)\). Since \(\text{Int}(A)\) is by definition the greatest open subset contained in \(A\), there exists some \(\varepsilon > 0\) such that \(B_{x,\varepsilon} \subseteq \text{Int}(A) \subseteq A\). Thus,

\[
B_{x,\varepsilon} \subseteq \bigcup \{U \text{ open}, U \subseteq A\}.
\]

Conversely, let \(y\) be an element in the union of all open subsets contained in \(A\). In particular, there exists such an open subset \(U\) containing \(y\), and by definition of \(\text{Int}(A)\) we have \(y \in U \subseteq \text{Int}(A)\).

\[\square\]

The following is an example of the importance of the discrete metric.

Example 3.19. Intuition from the Euclidean metric on the plane \(\mathbb{R}^2\) tells us that the closure of an open ball of radius \(\varepsilon > 0\) around an element \(x \in \mathbb{R}^2\) should be the subset of elements whose distance to \(x\) is smaller or equal than \(\varepsilon\):

\[
\text{Cl}(B_{x,\varepsilon}) = \{y \in \mathbb{R}^2 \mid d(x, y) \leq \varepsilon\}.
\]

This is false in general, and the discrete metric is the perfect counterexample. Indeed, let \(X\) be a set with the discrete metric, and let \(x \in X\). Recall from Example 3.13 that \(\{x\} = B_{x,1}\), the open ball of radius 1. Furthermore, \(\{x\}\) is both open and closed. Thus,

\[
\{x\} = B_{x,1} = \text{Cl}(B_{x,1}) \subseteq D_{x,1} = X.
\]
**Exercise 3.20.** Let $V$ be a normed real vector space, with norm $\rho$, and let $d$ be the induced metric. Prove that, in this case for any $v \in V$ and any $\varepsilon > 0$, the closure of the open ball of radius $\varepsilon$ around $v$ is the closed disk around $v$ of radius $\varepsilon$.

$$\text{Cl}(B_{v,\varepsilon}) = D_{v,\varepsilon}.$$ 

Let’s see some properties of interior, closure and boundary of subsets in a metric space.

**Lemma 3.21.** Let $(X, d)$ be a metric space. Then the following holds:

1. a subset $A \subseteq X$ is open if and only if $\text{Int}(A) = A$;
2. for every $A \subseteq X$, $\text{Int}(\text{Int}(A)) = \text{Int}(A)$;
3. if $\text{Int}(A) \subseteq B \subseteq A$ then $\text{Int}(A) = \text{Int}(B)$;
4. if $\{U_\alpha\}_{\alpha \in \Gamma}$ is a collection (possibly infinite) of subsets of $X$, then
   $$\bigcup_{\alpha \in \Gamma} \text{Int}(U_\alpha) \subseteq \text{Int}\left(\bigcup_{\alpha \in \Gamma} U_\alpha\right) \quad \text{and} \quad \text{Int}\left(\bigcap_{\alpha \in \Gamma} U_\alpha\right) \subseteq \bigcap_{\alpha \in \Gamma} \text{Int}(U_\alpha);$$
5. if $\{U_1, \ldots, U_n\}$ is a finite collection of subsets of $X$, then
   $$\text{Int}\left(\bigcap_{i=1}^n U_i\right) = \bigcap_{i=1}^n \text{Int}(U_i);$$

**Proof.** Properties (1) and (2) are obvious by definition. Property (3) is proved as follows. If $\text{Int}(A) \subseteq B$, then by definition $\text{Int}(A) \subseteq \text{Int}(B) \subseteq B \subseteq A$. Also, since $\text{Int}(B)$ is an open subset of $A$, it follows that $\text{Int}(B) \subseteq \text{Int}(A)$. Property (4) follows from the fact that, for each $\gamma \in \Gamma$, we have $\bigcap_{\Gamma} U_\alpha \subseteq U_\gamma \subseteq \bigcup_{\Gamma} U_\alpha$. Finally, let’s prove property (5). The intersection $I = \bigcap_{i=1}^n \text{Int}(U_i)$ is an open subset of $\bigcap_{i=1}^n U_i$, and hence $I \subseteq \text{Int}(\bigcap_{i=1}^n U_i)$. This combined with property (4) finishes the proof.

**Example 3.22.** This is an example about how the inclusion $\bigcup_{\Gamma} \text{Int}(U_\alpha) \subseteq \text{Int}\left(\bigcup_{\Gamma} U_\alpha\right)$ may fail to be an equality. Consider then $\mathbb{R}$ with the metric $d(x, y) = |x - y|$, and let $U_1 = [0, 1]$ and $U_2 = [1, 2]$. Then,

$$U = U_1 \cup U_2 = [0, 2] \quad \text{and} \quad \text{Int}(U) = (0, 2),$$

while $\text{Int}(U_1) = (0, 1)$ and $\text{Int}(U_2) = (1, 2)$. Notice that $\{1\} \notin (0, 2)$

$$\{1\} \notin (0, 1) \cup (1, 2) \not= (0, 2).$$

Note that, with the discrete metric, the inclusions in Lemma 3.21 (4) become equalities. Can you prove it?

A similar list of properties exists for the closure of a subset of a metric space. We state it below without proof (the proof is similar to the result above, so it is left as an exercise).
Lemma 3.23. Let \((X, d)\) be a metric space. Then the following holds:

1. A subset \(A \subseteq X\) is closed if and only if \(A = \text{Cl}(A)\);
2. For every \(A \subseteq X\), \(\text{Cl}(\text{Cl}(A)) = \text{Cl}(A)\);
3. If \(A \subseteq B \subseteq \text{Cl}(A)\) then \(\text{Cl}(A) = \text{Cl}(B)\);
4. If \(\{U_\alpha\}_{\alpha \in \Gamma}\) is a collection (possibly infinite) of subsets of \(X\), then
   \[ \bigcup_{\alpha \in \Gamma} \text{Cl}(U_\alpha) \subseteq \text{Cl} \left( \bigcup_{\alpha \in \Gamma} U_\alpha \right) \]
   \[ \bigcap_{\alpha \in \Gamma} \text{Cl}(U_\alpha) \subseteq \bigcap_{\alpha \in \Gamma} \text{Cl}(U_\alpha) \];
5. If \(\{U_1, \ldots, U_n\}\) is a finite collection of subsets of \(X\), then
   \[ \text{Cl} \left( \bigcup_{i=1}^n U_i \right) = \bigcup_{i=1}^n \text{Cl}(U_i) \];

Finally, let’s see some interesting properties of the interaction of interior, closure, and boundary.

Lemma 3.24. Let \((X, d)\) be a metric space. Then, for any subsets \(A, B \subseteq X\),

1. The closure of the complement of \(A\) is the complement of the interior of \(A\):
   \[ \text{Cl}(A^c) = \text{Cl}(X \setminus A) = X \setminus \text{Int}(A) = (\text{Int}(A))^c; \]
2. The interior of the complement of \(B\) is the complement of the closure of \(B\):
   \[ \text{Int}(B^c) = \text{Int}(X \setminus B) = X \setminus \text{Cl}(B) = (\text{Cl}(B))^c; \]
3. \(\partial(\text{Cl}(A)) \subseteq \text{Cl}(A)\).
4. \(\partial(\text{Int}(B)) \cap \text{Int}(B) = \emptyset; \) and

Proof. To prove (1), notice that \(\text{Int}(A) = A \setminus \partial(A) = A \cap (\partial(A))^c\). Thus,
   \[ (\text{Int}(A))^c = X \setminus \text{Int}(A) = X \setminus (A \cap (\partial(A))^c) = (X \setminus A) \cup (X \setminus (\partial(A))^c) = \]
   \[ = A^c \cup ((\partial(A))^c)^c = A^c \cup \partial(A) = A^c \cup \partial(A^c) = \text{Cl}(A^c) \]

To prove (2) just replace \(A\) by \(B^c\) everywhere above. Property (3) follows by
   \[ \partial(\text{Cl}(A)) = \text{Cl}(\text{Cl}(A)) \cap \text{Cl}(X \setminus \text{Cl}(A)) = \text{Cl}(A) \cap \text{Cl}(X \setminus \text{Cl}(A)) \subseteq \text{Cl}(A); \]

To prove (4), notice that \(Y \cap Y^c = \emptyset\) for any \(Y \subseteq X\). Thus, we just need to check that \(\partial(\text{Int}(B)) \subseteq (\text{Int}(B))^c\). Indeed,
   \[ \partial(\text{Int}(B)) = \partial((\text{Int}(B))^c) = \partial(\text{Cl}(B^c)) \subseteq \text{Cl}(B^c) = (\text{Int}(B))^c, \]

where (from left to right) the first equality holds by definition, the second equality holds by (2), the inequality holds by (3) and the last equality holds by (1). \(\square\)
Example 3.25. Consider the plane $\mathbb{R}^2$ with the Euclidean metric. For any point $u \in \mathbb{R}^2$, we have

$$\text{Int}\{\{u\}\} = \emptyset \quad \text{and} \quad \text{Cl}(\{u\}) = \{u\}.$$  
Thus, $\partial(\{u\}) = \{u\} \cap (\emptyset)^c = \{u\} \cap \mathbb{R}^2 = \{u\}.$

Still in the plane $\mathbb{R}^2$ with the Euclidean metric, we have

$$\text{Int}(B_{u,\varepsilon}) = B_{u,\varepsilon} \quad \text{and} \quad \text{Cl}(B_{u,\varepsilon}) = D_{u,\varepsilon},$$
and the boundary is then the circle of radius $\varepsilon$ and center $u$:

$$\partial(B_{u,\varepsilon}) = D_{u,\varepsilon} \cap (B_{u,\varepsilon})^c = \{v \in \mathbb{R}^2 \mid d(u, v) = \varepsilon\}.$$

If we consider now $\mathbb{R}^2$ with the discrete metric, then for each $u \in \mathbb{R}^2$,

$$\text{Int}(\{u\}) = \{u\} \quad \text{and} \quad \text{Cl}(\{u\}) = \{u\}$$
and thus $\partial(\{u\}) = \{u\} \cap (\{u\})^c = \emptyset$.

Exercise 3.26. Let $(X, d)$ be a metric space. Prove that, for any subset $A \subseteq X$,

$$\partial(A) = \text{Cl}(A) \setminus \text{Int}(A) \quad \text{def} = \{x \in \text{Cl}(A) \mid x \notin \text{Int}(A)\}.$$  
Deduce that, if $A$ is an open and closed subset of $X$, then $\partial(A) = \emptyset$.

3.3. Convergence of sequences in metric spaces. The idea behind the notion of limit of a sequence is that of approximating an object which is not completely known or too complicated by simpler objects. For instance, the value of the numbers $e$ and $\pi$ can be approximated by

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \quad \pi = 4 \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$
(notice that the sum of an series can be interpreted as a limit).

Limits of sequences of real numbers are just a particular case of a more general situation. For instance, recall from the introduction (Chapter 1) the example about Taylor approximation. In this section we will generalize the notion of sequence and the convergence of its limit to all metric spaces.

Definition 3.27. Let $(X, d)$ be a metric space and let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of elements of $X$. Then, the sequence $\{x_n\}_{n \in \mathbb{N}}$ converges to the element $x \in X$ if, for every $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that

$$d(x, x_n) < \varepsilon \quad \text{for all} \quad n \geq N.$$  
When this is the case, the element $x$ is called the limit of the sequence $\{x_n\}_{n \in \mathbb{N}}$, and we write $x = \lim_{n \to \infty} x_n$.

Remark 3.28. The definition of convergence of a sequence in a metric space can be rephrased in several equivalent ways, such as
(1) The sequence \( \{x_n\}_{n \in \mathbb{N}} \) converges to \( v \) if the sequence of numbers \( \{d(v, x_n)\}_{n \in \mathbb{N}} \) converges to zero.

(2) The sequence \( \{x_n\}_{n \in \mathbb{N}} \) converges to \( v \) if for every \( \varepsilon > 0 \) the open ball \( B_{v, \varepsilon} \) contains all but a finite number of terms of the sequence.

\[ \begin{array}{c}
  x_1 \quad x_2 \quad \cdots \quad x_4 \\
  \phantom{x_1} \phantom{x_2} \quad \cdots \\
  x_3 \\
\end{array} \]

\[ B_{v, \varepsilon} \]

**Theorem 3.29.** Let \( (X, d) \) be a metric space. If the sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \( X \) has a limit, then it is unique.

**Proof.** Suppose the sequence \( \{x_n\}_{n \in \mathbb{N}} \) converges to the points \( x, y \in X \), and suppose \( x \neq y \). This means that, for each \( \varepsilon > 0 \) there exist \( N_x, N_y \in \mathbb{N} \) such that

\[ d(x, x_n) < \varepsilon, \text{ for all } n \geq N_x \quad \text{and} \quad d(y, x_n) < \varepsilon, \text{ for all } n \geq N_y. \]

In particular, let \( \varepsilon = \frac{d(x, y)}{2} > 0 \), and let \( N = \max\{N_x, N_y\} \), where \( N_x, N_y \) are as above (with respect to this particular choice of \( \varepsilon \)). Then, for each \( n \geq N \), we have \( d(x, x_n), d(y, x_n) < \varepsilon \). On the other hand, by the triangle inequality,

\[ d(x, y) \leq d(x, x_n) + d(x_n, y) < 2 \cdot \varepsilon = 2 \cdot \frac{d(x, y)}{2} = d(x, y), \]

which is impossible. Thus, \( x = y \). \( \square \)

**Example 3.30.** Let \( X \) be a set with the discrete metric. Then, a sequence \( \{x_n\}_{n \in \mathbb{N}} \) is convergent if and only if there exists some \( N \in \mathbb{N} \) such that \( x_n = x_N \) for all \( n \geq N \). Studying the convergence of sequences in a set with the discrete metric is not really interesting.

**Example 3.31.** Let \( X \) be the set of all continuous functions \( f : [0, 1] \to \mathbb{R} \), and consider the following two metrics on \( X \):

\[ d_{\text{int}}(f, g) = \int_0^1 |f(x) - g(x)| \, dx \quad d_{\infty}(f, g) = \max\{|f(x) - g(x)| \mid x \in [0, 1]\}. \]

Now fix some \( \alpha > 0 \) and let \( \{f_n\}_{n \in \mathbb{N}} \) be the sequence where \( f_n \) is the function with graph

\[ (0, \alpha) \]

\[ (0, 0) \]

\[ (\frac{1}{2} - \frac{1}{n}, 0) \quad (\frac{1}{2}, 0) \quad (\frac{1}{2} + \frac{1}{n}, 0) \]

\[ (1, 0) \]
Then, the sequence \( \{f_n\}_{n \in \mathbb{N}} \) converges to the function \( g = 0 \) (constant on \([0, 1]\) with image 0) with respect to the metric \( d_{\text{int}} \). Indeed, for each \( n \)
\[
d_{\text{int}}(0, f_n) = \int_0^1 |0 - f_n(x)| \, dx = \int_0^1 f_n(x) \, dx = \frac{1}{n}
\]
and
\[
\lim_{n \to \infty} \frac{1}{n} = 0.
\]
However, the same sequence \( \{f_n\} \) does not converge to the constant function \( g = 0 \) with respect to the metric \( d_\infty \), since for any \( n \)
\[
d_\infty(0, f_n) = \max\{|0 - f_n(x)| \mid x \in [0, 1]\} = \alpha > 0.
\]
In fact, this sequence does not converge to any continuous function with respect to the metric \( d_\infty \), but we are not ready yet to prove this.

Sequences in a metric space provide a nice criterion to detect closed subsets.

**Theorem 3.32.** Let \((X, d)\) be a metric space and let \( V \subseteq X \) be a subset. Then the following statements are equivalent.

1. \( V \) is a closed subset of \( X \).
2. The limit of every convergent sequence \( \{x_n\}_{n \in \mathbb{N}} \subseteq V \) is an element of \( V \).

**Proof.** Suppose first that (2) is true. We prove by contradiction that \( V^c \) is open. Assume then that \( V^c \) is not open, and let \( u \in V^c \) be an element for which \( B_{u, \varepsilon} \not\subseteq V^c \), for any \( \varepsilon > 0 \). This means, in particular, that for any \( n \in \mathbb{N} \) the ball \( B_{u, \frac{1}{n}} \) contains (at least) an element \( x_n \) of \( V \). We can form then the sequence \( \{x_n\}_{n \in \mathbb{N}} \subseteq V \), but this sequence clearly converges to the element \( u \notin V \), hence the contradiction.

Suppose now that (1) is true. We have to show that every convergent sequence of elements of \( V \) has its limit in \( V \). Let then \( \{x_n\}_{n \in \mathbb{N}} \subseteq V \) be a convergent sequence, and suppose \( v = \lim x_n \in V^c \). Since \( V^c \) is open by hypothesis, there exists some \( \varepsilon > 0 \) such that \( B_{v, \varepsilon} \subseteq V^c \). On the other hand, since \( v = \lim x_n \), there exists some \( \varepsilon' < \varepsilon \) and some \( N \in \mathbb{N} \) such that \( d(v, x_n) < \varepsilon' \) for all \( n \geq N \), and this implies that
\[
x_n \in B_{v, \varepsilon'} \subseteq B_{v, \varepsilon} \subseteq V^c,
\]
which contradicts the assumption that \( x_n \in V \) for all \( n \).

We already know examples of different metrics on a set \( X \) which define the same collection of open subsets of \( X \). Thus we should not be surprised by the following result.

**Corollary 3.33.** Let \( X \) be a set and let \( d, d' \) be two topologically equivalent metrics on \( X \). Then, a sequence \( \{x_n\}_{n \in \mathbb{N}} \) converges to \( x \) with respect to \( d \) if and only if it converges to \( x \) with respect to \( d' \).
4. Continuous functions between metric spaces

We are finally ready to define and study continuous functions. Essentially, a function between sets \( f : X \to Y \) is just a relationship that we establish between the sets \( X \) and \( Y \), usually in order to express events happening in \( Y \) in terms of the elements in \( X \).

Sometimes we happen to have a deeper understanding of the sets \( X \) and \( Y \), which translates into a richer, common structure, and then we want the function \( f \) to take into account such structure. For instance, \( X \) and \( Y \) could be groups, or rings, or algebras over a given ring, and then we may want \( f \) to be a group homomorphism, or a ring homomorphism, or an algebra homomorphism.

And the same idea applies if we are dealing with metric spaces. That is, if \((X, d_X)\) and \((Y, d_Y)\) are metric spaces, we do not want to consider any map between the sets \( X \) and \( Y \), but maps that take into consideration the corresponding metrics. So, what about considering functions \( f : X \to Y \) satisfying \( d_Y(f(x), f(x')) = d_X(x, x') \) for any \( x, x' \in X \)?

**Definition 4.1.** Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. A map \( f : X \to Y \) such that
\[
d_Y(f(x), f(x')) = d_X(x, x')
\]
for all \( x, x' \in X \) is called an **isometry between** \( X \) and \( Y \).

We have encountered isometries already in these notes: the maps \( i_w : X \to X \times Y \) and \( j_z : Y \to X \times Y \) in Theorem 3.9 are isometries (check it!).

**Example 4.2.** Consider the sets \( \mathbb{R}^2 \) and \( \mathbb{C} \), both with the Euclidean metric. Then, the map \( f : \mathbb{C} \to \mathbb{R}^2 \) defined by \( f(a + bi) = (a, b) \) is an isometry.

It is clear that the condition on a map to be an isometry is rather restrictive. For instance, the map \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = x^2 \) is not an isometry:
\[
d(3, 4) = \sqrt{9 + 16} = 5 \text{ but } d(f(3), f(4)) = d(9, 16) = \sqrt{81 + 256} = \sqrt{337}.
\]

We want to consider more maps between metric spaces than just isometries, and the key lies in the definition of continuity for functions \( f : \mathbb{R} \to \mathbb{R} \). Before stating the proper, rigorous definition of continuity of such a function, think of the idea behind continuity:

\( f \) is continuous at \( a \in \mathbb{R} \) if \( f(x) \) is “close” to \( f(a) \) whenever \( x \in \mathbb{R} \) is “close” to \( a \).

This is what we want for functions between metric spaces to take into consideration.

The formalization of such idea is easy once we formalize continuity for functions \( f : \mathbb{R} \to \mathbb{R} \): recall that such a function is **continuous at** \( a \in \mathbb{R} \) if, for all \( \epsilon > 0 \) there exists some \( \delta > 0 \) such that
\[
\text{if } |a - x| < \delta \text{ then } |f(a) - f(x)| < \epsilon.
\]
Well, it is obvious now how to extend this definition to a map between metric spaces.

**Definition 4.3.** Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces, and let \(f : X \to Y\) be a set map. The map \(f\) is **continuous at the point** \(a \in X\) (with respect to the metrics \(d_X, d_Y\)) if for each \(\varepsilon > 0\) there exists some \(\delta > 0\) such that

\[
\text{if } d_X(a, x) < \delta \text{ then } d_Y(f(a), f(x)) < \varepsilon.
\]

If \(f\) is continuous at every element \(a \in X\), then we say that \(f\) is **continuous**.

Notice that the above definition depends on the given metrics \(d_X\) and \(d_Y\). In order to specify that the map \(f\) is continuous with respect to the metrics \(d_X\) and \(d_Y\) on \(X\) and \(Y\) respectively, we may also write

\[
f : (X, d_X) \to (Y, d_Y).
\]

This is actually rather important since the map \(f : X \to Y\) can be continuous with a choice of metrics and discontinuous with another choice of metrics. Let’s see this with an example.

**Example 4.4.** Let \(X = \mathbb{R} = Y\), and let \(f : \mathbb{R} \to \mathbb{R}\) be the map

\[
f(x) = \begin{cases} 
0, & \text{if } x \in \mathbb{Q} \\
1, & \text{if } x \notin \mathbb{Q}
\end{cases}
\]

Let then \(d_{\text{Euc}}\) and \(d_{\text{dis}}\) be the Euclidean and the discrete metrics on \(\mathbb{R}\). We have then

- \(f : (X, d_{\text{Euc}}) \to (Y, d_{\text{Euc}})\) is not continuous;
- \(f : (X, d_{\text{Euc}}) \to (Y, d_{\text{dis}})\) is not continuous;
- \(f : (X, d_{\text{dis}}) \to (Y, d_{\text{Euc}})\) is continuous; and
- \(f : (X, d_{\text{dis}}) \to (Y, d_{\text{dis}})\) is continuous.

Of course, this is an extreme example, but it perfectly describes the importance of keeping in mind which are the ambient metrics. As an exercise, fill in the details to show each of the points above.

**Remark 4.5.** The discrete metric also illustrates the following (not intuitive) situation: a bijective map \(f : X \to Y\) of sets which is continuous with respect to a choice of metrics \(d_X\) and \(d_Y\) may not have a continuous inverse! Indeed, the identity map \((\mathbb{R}, d_{\text{dis}}) \to (\mathbb{R}, d_{\text{Euc}})\) is continuous, but the inverse is not.

Let’s see some more examples of continuous functions.

**Exercise 4.6.** Let \(f : (X, d_X) \to (Y, d_Y)\) be an isometry. Prove that

1. \(f\) is injective (as a map of sets); and
2. \(f\) is a continuous function (with respect to \(d_X\) and \(d_Y\)).
Example 4.7. The map \( \mu : \mathbb{R}^2 \rightarrow \mathbb{R} \) defined by \( \mu(a, b) = a \cdot b \) (multiplication of real numbers) is a continuous function with respect to Euclidean metrics on \( \mathbb{R}^2 \) and \( \mathbb{R} \), which we will denote here by \( d_2 \) and \( d_1 \) respectively to avoid confusion.

Fix then some \((a, b) \in \mathbb{R}^2\). To prove that \( \mu \) is continuous at \((a, b)\) we have to check that for each \( \varepsilon > 0 \) there exists some \( \delta > 0 \) such that

\[
\text{if } d_2((a, b), (u, v)) < \delta \text{ then } d_1(\mu(a, b), \mu(u, v)) = d_1(ab, uv) < \varepsilon.
\]

Let’s ignore \( \varepsilon \) for a second, and suppose that \( d_2((a, b), (u, v)) = \sqrt{(a - u)^2 + (b - v)^2} < \delta \) for some \( \delta > 0 \). Then, we have

\[
|a - u|, |b - v| < \delta,
\]

and

\[
d_1(ab, uv) = |ab - uv| = |b(a - u) + a(b - v) + (a - u)(v - b)| \leq |b| \cdot |a - u| + |a| \cdot |b - v| + |a - u| \cdot |b - v| < \delta^2 + \delta(|a| + |b|).
\]

Let’s go back now to the question of the continuity of \( \mu \) at \((a, b) \in \mathbb{R}^2\). Let then \( \varepsilon > 0 \), and consider the quadratic equation on \( \delta \):

\[
\delta^2 + (|a| + |b|)\delta - \varepsilon = 0.
\]

This equation has always a positive solutions, namely \( \delta = \frac{-(|a| + |b|) + \sqrt{(|a| + |b|)^2 + 4\varepsilon}}{2} \), and the above discussion implies that, for this particular choice of \( \delta \), if \( d_2((a, b), (u, v)) < \delta \) then \( d_1(ab, uv) < \varepsilon \) as required.

Exercise 4.8. Let \((X, d)\) be a metric space and let \( z \in X \). We can then define a map

\[
X \xrightarrow{\delta_z} [0, \infty),
\]

\[
x \mapsto \delta_z(x) = d(z, x)
\]

Prove that the map \( \delta_z \) is continuous with respect to the metric \( d \) on \( X \) and the Euclidean metric on \([0, \infty) \subseteq \mathbb{R}\).

Example 4.9. Let \( d \) be the \( p \)-adic metric on the set of integers \( \mathbb{Z} \), and fix some \( \alpha \in \mathbb{Z} \). Then, the maps

\[
\begin{align*}
(Z, d) \xrightarrow{f} (Z, d) & \quad (Z, d) \xrightarrow{g} (Z, d) \\
k & \mapsto \alpha \cdot k & k & \mapsto k + \alpha
\end{align*}
\]

are continuous.

Let’s prove first that \( f \) is continuous. For every \( a, b \in \mathbb{Z} \), we have

\[
d(f(a), f(b)) = d(\alpha \cdot a, \alpha \cdot b) = \frac{1}{p^{\nu_p(a - \alpha b)}} = \frac{1}{p^{\nu_p(\alpha(a - b))}} = \frac{1}{p^{\nu_p(\alpha)} \cdot p^{\nu_p(a - b)}} = \frac{1}{p^{\nu_p(\alpha)}} \cdot d(a, b),
\]

where \( \nu_p(\cdot) \) denotes the \( p \)-adic valuation.
Since \( \frac{1}{p^n(n)} \leq 1 \), it follows that, for each \( a \in \mathbb{Z} \) and each \( \varepsilon > 0 \),

\[
\text{if } d(a, b) < \varepsilon \text{ then } d(f(a), f(b)) \leq d(a, b) < \varepsilon
\]

and hence \( f \) is a continuous map. In particular notice that if \( p \) does not divide \( \alpha \) then \( f \) is an isometry.

Let’s check now that the map \( g \) is continuous. In this case, for each \( a, b \in \mathbb{Z} \) we have

\[
d(f(a), f(b)) = d(a + \alpha, b + \alpha) = \frac{1}{p^{n(a+b-(b+a))}} = \frac{1}{p^{n(a-b)}} = d(a, b),
\]

so in fact this map is always an isometry (in particular, it is continuous).

Once we start working with functions, it is natural to consider compositions of functions.

**Proposition 4.10.** Let \( f: (X, d_X) \to (Y, d_Y) \) and \( g: (Y, d_Y) \to (Z, d_Z) \) be continuous maps of metric spaces. Then, the composition map, \( (g \circ f): (X, d_X) \to (Z, d_Z) \), is a continuous map of metric spaces.

**Proof.** Set for simplicity \( h = (g \circ f): X \to Z \), and fix an element \( a \in X \). We have to check that, for each \( \varepsilon > 0 \) there is some \( \delta > 0 \) such that, if \( d_X(a, x) < \delta \), then \( d_Z(h(a), h(x)) < \varepsilon \).

Let \( b = f(a) \in Y \). Since \( g \) is continuous, there exists some \( \gamma > 0 \) such that, if \( d_Y(b, y) < \gamma \), then

\[
d_Z(g(b), g(y)) = d_Z(h(a), g(y)) < \varepsilon.
\]

Similarly, since \( f \) is continuous, there exists some \( \delta > 0 \) such that, if \( d_X(a, x) < \delta \), then

\[
d_Y(f(a), f(x)) = d_Y(b, f(x)) < \gamma.
\]

Combining the above, we get the following:

\[
d_X(a, x) < \delta \implies d_Y(f(a), f(x)) = d_Y(b, f(x)) < \gamma \implies
d_Z(g(b), g(f(x))) = d_Z(h(a), h(x)) < \varepsilon
\]

which proves that the composition is continuous at \( a \in X \). \( \square \)

Let’s analyze the definition of continuity. Let then \( f: (X, d_X) \to (Y, d_Y) \) be a continuous function, and let \( a \in X \). The map \( f \) is continuous at \( a \) if, for each \( \varepsilon > 0 \) there is some \( \delta > 0 \) such that

\[
\text{if } d_X(a, x) < \delta \text{ then } d_Y(f(a), f(x)) < \varepsilon.
\]

The left part above says that \( x \in B_{a,\delta} \), while the right part says that \( f(x) \in B_{f(a),\varepsilon} \).

This suggests some relation between continuity and open subsets, which we formulate properly as the following result. You can take it as a set of equivalent definitions of the notion of continuity.
**Theorem 4.11.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, and let $f : X \to Y$ be a map. Then, the following statements are equivalent:

1. $f$ is a continuous function;
2. for each open subset $U \subseteq Y$, $f^{-1}(U)$ is an open subset of $X$;
3. for each closed subset $V \subseteq Y$, $f^{-1}(V)$ is a closed subset of $X$;
4. for any sequence $\{x_n\}_{n \in \mathbb{N}}$ in $X$, if $\{x_n\}$ converges to $z \in X$, then the sequence $\{f(x_n)\}_{n \in \mathbb{N}}$ in $Y$ converges to $f(z)$.

A note of warning: given a map of sets $f : X \to Y$, the notation $f^{-1}(U)$ stands simply for the set of pre-images in $X$ of the elements of $U$.

We are not assuming the existence of any inverse function! For instance consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Clearly, this function does not have an inverse, but the pre-image of the subset $\{1\} \in \mathbb{R}$ makes perfect sense:

$$f^{-1}(\{1\}) = \{x \in \mathbb{R} \mid f(x) = 1\} = \{-1, 1\}.$$

**Remark 4.12.** With this set of equivalent definitions of continuity, we do not need to define continuity at a point, we can define continuous functions directly: a function between metric spaces $f : X \to Y$ is continuous if and only if for each open subset $U \subseteq Y$ the pre-image $f^{-1}(U)$ is an open subset of $X$.

Be careful with the following: condition (2) above concerns only the pre-image of open subsets of $Y$, but it says nothing about the image of open subsets of $X$. The following is, in general, false:

if $U \subseteq X$ is an open subset, then $f(U) \subseteq Y$ is an open subset.

**Proof of Theorem 4.11.** Since this proof is rather long, let’s split it in smaller parts, each proving an equivalence of statements.

- Statements (1) and (2) are equivalent.

First we prove that (1) implies (2). Suppose then that $f$ is a continuous map, and let $U \subseteq Y$. Let also $a \in f^{-1}(U)$, and let $u = f(a)$. Since $U$ is open, there exists some $\varepsilon(u) > 0$ such that $B_{u,\varepsilon(u)} \subseteq U$. Also, since $f$ is continuous, there exists some $\delta(a) > 0$ such that

if $d_X(a, x) < \delta(a)$ then $d_Y(f(a), f(x)) = d_Y(u, f(x)) < \varepsilon(u),$

which implies that $f(B_{a,\delta(a)}) \subseteq U$, and hence $B_{a,\delta(a)} \subseteq f^{-1}(U)$. This proves that $f^{-1}(U)$ is open with respect to $d_X$.

Let’s see now that (2) implies (1). Suppose then that $f^{-1}(U)$ is an open subset of $X$ for each open subset $U \subseteq Y$, and fix $a \in X$. For each $\varepsilon > 0$, the open ball $B_{f(a),\varepsilon}$ is
an open subset of \( Y \), and thus \( f^{-1}(B_{f(a),\varepsilon}) \) is an open subset of \( X \). Furthermore,
\[
a \in f^{-1}(B_{f(a),\varepsilon}),
\]
and this implies that there exists some \( \delta > 0 \) such that
\[
B_{a,\delta} \subseteq f^{-1}(B_{f(a),\varepsilon}).
\]
The above inclusion is equivalent to say that \( f \) is continuous at \( a \in X \).

- Statements (2) and (3) are equivalent.

Let \( V \subseteq Y \) be a closed subset, and let \( U = V^c \), which is an open subset by hypothesis. By (2), \( f^{-1}(U) \) is an open subset of \( X \). Furthermore, we have
\[
f^{-1}(U) = f^{-1}(V^c) = f^{-1}(Y \setminus V) = X \setminus f^{-1}(V),
\]
and thus \( f^{-1}(V) \) is a closed subset, and this proves that (2) implies (3). Replacing “open” by “closed” proves the converse.

- Statements (1) and (4) are equivalent.

First we prove that (1) implies (4). Let then \( \{x_n\}_{n \in \mathbb{N}} \) be a sequence in \( X \) which converges to \( z \in X \). Since \( f \) is continuous (at \( f(z) \)), for each \( \varepsilon > 0 \) there exists some \( \delta > 0 \) such that
\[
\text{if } d_X(z, x) < \delta \text{ then } d_Y(f(z), f(x)) < \varepsilon.
\]
Also, since the sequence is convergent, we know that there exists some \( N \in \mathbb{N} \) such that \( d_X(z, x_n) < \delta \) for all \( n \geq N \). Thus, for all \( n \geq N \), we have
\[
d_Y(f(z), f(x_n)) < \varepsilon,
\]
which implies (4).

Finally, we have to check that (4) implies (1). Suppose otherwise that (1) is not true on a fixed element \( a \in X \): there exists some \( \varepsilon > 0 \) such that for each \( \delta > 0 \) there is some \( x \in X \) such that
\[
d_X(a, x) < \delta \text{ but } d_Y(f(a), f(x)) \geq \varepsilon.
\]
By fixing such \( \varepsilon \), we can then define the sets
\[
A_n = \{ x \in X \mid d_X(a, x) < \frac{1}{n} \text{ and } d_Y(f(a), f(x)) \geq \varepsilon \},
\]
which is non-empty for all \( n \in \mathbb{N} \). Next choose for each \( n \) an element \( x_n \in A_n \): the sequence \( \{x_n\}_{n \in \mathbb{N}} \) clearly converges to \( a \in X \), while the sequence \( \{f(x_n)\}_{n \in \mathbb{N}} \) does not converge to \( f(a) \), contradicting (4).

**Exercise 4.13.** At the beginning of this chapter we saw several examples of continuous functions. Which of them are easier to check with these alternative definitions?

**Exercise 4.14.** Let \( f : X \to Y \) be a map. Prove that \( f \) is continuous with respect to the metrics \( d_X \) and \( d_Y \) (on \( X \) and \( Y \) respectively) if and only if, for each \( y \in Y \) and each \( \varepsilon > 0 \),
\[
f^{-1}(B_{y,\varepsilon}^{d_Y}) \text{ is an open subset of } X.
\]
In other words, it is enough to check condition (2) in Theorem 4.11 for all the open balls in $Y$.

A first consequence of Theorem 4.11 is that topologically equivalent metrics define the same continuous maps. The following result state this idea rigorously.

**Corollary 4.15.** Let $f : X \to Y$ be a map. Let also $d_X, d'_X$ be topologically equivalent metrics on $X$ and $d_Y, d'_Y$ be topologically equivalent metrics on $Y$. Then, the following statements are equivalent:

1. the map $f$ is continuous with respect to the metrics $d_X$ and $d_Y$; and
2. the map $f$ is continuous with respect to the metrics $d'_X$ and $d'_Y$.

Conversely, suppose that, for every map $f : X \to Y$, $f$ is continuous with respect to the metrics $d_X$ and $d_Y$ if and only if $f$ is continuous with respect to the metrics $d'_X$ and $d'_Y$. Then, the following holds:

1. the metrics $d_X$ and $d'_X$ are topologically equivalent; and
2. the metrics $d_Y$ and $d'_Y$ are topologically equivalent.

**Proof.** Since topologically equivalent metrics define the same collections of open sub-sets, the result follows immediately by using (2) in Theorem 4.11.

**Exercise 4.16.** Prove the above result using only the epsilon-delta definition of continuity. Do not expect the proof to be shorter than the above!

#### 4.1. Homeomorphisms of metric spaces and open maps.

We have see in Remark 4.5 how a bijective map of sets, which is in addition a continuous map between metric spaces, does not have a continuous inverse. This section compiles a few useful notions related to the aforementioned remark, as well as some interesting examples or exercises.

**Definition 4.17.** Let $f : (X, d_X) \to (Y, d_Y)$ be a map of sets. Then, $f$ is a **homeomorphism** if the inverse map $f^{-1}$ exists and both $f$ and $f^{-1}$ are continuous. Similarly, two metric spaces $(X, d_X)$ and $(Y, d_Y)$ are **homeomorphic** if there exists a homeomorphism $f : X \to Y$.

Broadly speaking, the notion of homeomorphism is to metric spaces as the notion of isomorphism is to groups or other algebraic structures. Notice that the condition of $f^{-1}$ existing is already rather restrictive: you know of many continuous maps $f : \mathbb{R} \to \mathbb{R}$ which do not have an inverse function.

**Example 4.18.** Recall from Remark 4.12 that a continuous function $f : X \to Y$ does not necessarily map open subsets of $X$ to open subsets of $Y$. As usual, a
counterexample to this statement arises from the discrete metric. Recall from Remark 4.5 that the map
\[ \text{Id}: (\mathbb{R}, d_{\text{dis}}) \rightarrow (\mathbb{R}, d_{\text{Euc}}) \]
is continuous. Furthermore, we know that for any \( x \in \mathbb{R} \), the singleton \( \{x\} \subseteq \mathbb{R} \) is an open subset with respect to the discrete metric. On the other hand we have \( \text{Id}(\{x\}) = \{x\} \subseteq \mathbb{R} \), which is not an open subset with respect to the Euclidean metric.

**Definition 4.19.** A map \( f: (X, d_X) \rightarrow (Y, d_Y) \) is open (respectively, closed) if \( f(U) \) is open (respectively, closed) for each open (respectively, closed) subset \( U \subseteq X \).

Notice that, in the definition above, we do not require \( f \) to be continuous! For example the map \( \text{Id}: (\mathbb{R}, d_{\text{Euc}}) \rightarrow (\mathbb{R}, d_{\text{dis}}) \) is open but not continuous.

**Example 4.20.** The following example shows that open maps need not be closed, and vice-versa. For instance, let \( X \) be a finite set with the discrete metric. We can then list the elements of \( X \), namely \( X = \{x_1, \ldots, x_n\} \), and then define a map \( f \) from \( X \) with the discrete metric to \( \mathbb{R} \) with the Euclidean metric by
\[ f(x_k) = k \text{ for } k = 1, \ldots, n. \]
Since \( X \) has the discrete metric it is immediate that this map is continuous (although such property is independent from \( f \) being open, closed, both or none).

Let’s check that this map is closed. First of all, every subset \( \{x_k\} \subseteq X \) is mapped to \( \{k\} \subseteq \mathbb{R} \), which is also a closed subset (see Exercise 3.15). Since \( X \) is finite, any closed subset \( V \) is a finite union of singletons, and thus so is \( f(V) \). Hence, \( f(V) \) is closed in \( \mathbb{R} \) with the discrete metric. However, this map is clearly not open (see Example 4.18 above).

Let now \( X = [0, 1] \), with the metric induced from \( \mathbb{R} \), and let \( f: X \rightarrow X \) be defined as follows
\[ f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2} \\ x & \text{if } \frac{1}{2} < x \leq 1 \end{cases} \]
This map is not continuous at \( x = \frac{1}{2} \), but this is not relevant for the example. What matters in this example is that it is not closed, since \( f\left(\left[\frac{1}{2}, 1\right]\right) = \left(\frac{1}{2}, 1\right] \), but it is open. Indeed, let \( x \in X \) and \( \varepsilon > 0 \). Then,
- if \( x \in \left[0, \frac{1}{2}\right) \) and \( \frac{1}{2} \notin B_{x, \varepsilon} \), then \( f\left(B_{x, \varepsilon}\right) = B_{2x, 2\varepsilon} \);
- if \( x \in \left[0, \frac{1}{2}\right) \) and \( \frac{1}{2} \in B_{x, \varepsilon} \), then set \( x_1 = \frac{2x+2\varepsilon+1}{4} \) and \( \delta_1 = \frac{2x+2\varepsilon-1}{4} \), and
  \[ f\left(B_{x, \varepsilon}\right) = (2(x - \varepsilon), 1] \cup B_{x_1, \delta_1} \];
- if \( x = \frac{1}{2} \), then \( f\left(B_{\frac{1}{2}, \varepsilon}\right) = \left(\frac{1}{2}, 1]\cup \left(\frac{1}{2}, \frac{1}{2} + \varepsilon\right) \);
- if \( x \in (\frac{1}{2}, 1] \) and \( \frac{1}{2} \in B_{x, \varepsilon} \), then \( f\left(B_{x, \varepsilon}\right) = (1 - x + \varepsilon, 1] \cup (\frac{1}{2}, x + \varepsilon) \); and
- if \( x \in (\frac{1}{2}, 1] \) and \( \frac{1}{2} \notin B_{x, \varepsilon} \), then \( f\left(B_{x, \varepsilon}\right) = B_{x, \varepsilon} \).
Exercise 4.21. Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces, and consider \(X \times Y\) with the product metric. Then, the projection maps \(\pi_X : X \times Y \to X\) and \(\pi_Y : X \times Y \to Y\) are continuous and open maps. Can you give an example of such a projection map which is not closed?

Theorem 4.22. Let \(f : (X, d_X) \to (Y, d_Y)\) be a map. Then, the following are equivalent:

1. \(f\) is a homeomorphism;
2. \(f\) is bijective, continuous and open;
3. \(f\) is bijective, continuous and closed;

In particular, every homeomorphism is an open and closed map.

Proof. Suppose first that \(f\) is a homeomorphism. Then \(f\) is continuous and bijective, and we have to check that it is open (and closed). Since \(f\) is a homeomorphism, we know that the map \(g = f^{-1}\) is continuous. Then, for each open (closed) subset \(U \subseteq X\), we have \(f(U) = g^{-1}(U)\), which is open (closed) since \(g\) is a continuous map. Thus, \(f\) is open (closed), and (1) implies (2) and (3).

Suppose now that \(f\) is bijective, continuous and open (closed). In particular, since \(f\) is bijective, we only have to check that the inverse map \(g = f^{-1}\) is continuous, i.e., that \(g^{-1}(U)\) is open (closed) for every open (closed) subset \(U \subseteq X\). But this is immediate: \(g^{-1}(U) = f(U)\) is open (closed) since \(f\) is an open (closed) map. Thus, (2) implies (1) (respectively (3) implies (1)).

Exercise 4.23. Consider \(\mathbb{R}^2\) with the Euclidean metric, and let

\[
S^1 = \{(x, y) \mid x^2 + y^2 = 1\} \subseteq \mathbb{R}^2
\]

be the circle of radius 1 around the origin. Considering then \(S^1\) and \(D_{(0,0),1}\) (the disk of radius 1 around the point \((0,0)\)) as metric subspaces of \(\mathbb{R}^2\), prove the following statement, known as Alexander’s trick: every homeomorphism \(f : S^1 \to S^1\) can be extended to a homeomorphism of the disk of radius 1, \(D_{0,1}\). Can you do the same in higher dimensions?

Exercise 4.24. Consider \(X = (0, 1)\) and \(Y = (0, \infty)\) as metric subspaces of \(\mathbb{R}\), and prove that \(X\) and \(Y\) are homeomorphic but not isometric.

Exercise 4.25. Let \(f : (X, d_X) \to (Y, d_Y)\) be a continuous map, and let

\[
G_f = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y
\]

be the graph of the function. Prove that, as a metric subspace of \(X \times Y\) (with the product metric), \(G_f\) is homeomorphic to \(X\).

Now that we know about continuous functions, we want to study how different properties of metric spaces behave with respect to continuous maps.
5. Interlude

During the previous chapters several notions have been introduced which were probably new to you, and even some of them blatantly defied our previous intuition. A lot of work needed to be done in order to give the theory of metric spaces consistency (recall the question of defining a suitable metric for a product of metric spaces).

So, where is all this leading too? Why going through all the trouble? This interlude intends to partially answer this, with an absolute lack of rigor. The only purpose of this chapter is to motivate all the work we have done until this point and the work we will do in the next chapters. Do not take anything in this chapter as rigorous!

Actually, there are many reasons to study metric and topological spaces. For instance, we have already seen in the tutorials that all circles in the Euclidean plane are homeomorphic to the circle of radius 1 and center (0, 0), and a similar statement is true for all disks in the Euclidean plane, and like this for many other geometric objects in $\mathbb{R}^2$ (and not only in $\mathbb{R}^2$ but in $\mathbb{R}^n$ for any $n \geq 1$). This is simply a matter of form or shape, but not of size.

Let's formalize this. Consider $\mathbb{R}^n$ with the Euclidean metric, and call a subset $U \subseteq \mathbb{R}^n$ bounded if there exist $x \in \mathbb{R}^n$ and $\varepsilon > 0$ such that $U \subseteq B_{x,\varepsilon}$. We can define an equivalence relation among all bounded subsets of $\mathbb{R}^n$: $U$ is equivalent to $V$ if they are homeomorphic (check that this is an equivalence relation). And now we can say that a shape is a class under this equivalence relation.

For instance you may have heard of the following example already: a mug and a doughnut have the same shape. Making the homeomorphism explicit can be a bit complicated, but here is an outline of how to go from the doughnut to the mug.

So we need to find ways of telling one shape from another: properties of metric spaces which do not change under homeomorphism, and which we can use to distinguish different shaper. For instance, a cylinder and a Moebius band are the same shape? Let’s see. We can represent each of them as follows

where the arrows on each side of the bands determine how to glue one side with the other. What happens then with the boundary of the cylinder? It is formed by two
disjoint circles. And the boundary of the Moebius band? It is formed by a single
circle (well, or something that resembles a circle), so here is something: cylinder and
Moebius band cannot be the same shape if the boundaries are so different.

Related to the above discussion is another problem which boils down to metric (and
topological) spaces. Finding solutions to polynomial functions can get pretty difficult
if the polynomial has more than one variable and the degree is high. For instance,
what about solving the equation
\[ 5x^2y^3 - xy^4 + y^5 - x^5 = 0? \]

Instead, let’s give this problem a different approach. Let \( P: \mathbb{R}^n \to \mathbb{R} \) be a polynomial
function (in one or more variables). The set
\[ P^{-1}(\{0\}) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid P(x_1, \ldots, x_n) = 0\} \subseteq \mathbb{R}^n \]
determines a metric subspace of \( \mathbb{R}^n \). For instance, for the polynomial function
\( P(x,y) = x^2 + y^2 - 1 \) we get
\[ P^{-1}(\{0\}) = S^1, \]
or, for something a bit fancier, consider \( P(x,y,z) = x^2 + y^2 + z^2 - 1 \) and \( Q(x,y,z) = (2 - \sqrt{x^2 + y^2})^2 + z^2 - 1 \). Then, \( P^{-1}(\{0\}) \) is a sphere, while \( Q^{-1}(\{0\}) \) is a torus:

![Sphere](image1.png) ![Torus](image2.png)

Understanding the shape of \( P^{-1}(\{0\}) \) helps finding solutions or discarding the exis-
tence of them. Actually, believe or not, the study of such subsets leads to the \( p \)-adic
metric (but this is rather complicated to fit in this course, a shame).

The above example connects with the following situation, in the sense that again
polynomial functions are involved. Consider the continuous function \( f: \mathbb{R} \to \mathbb{R} \) given
by \( f(x) = x^2+1 \). We know that, as such, there is no solution to the equation \( f(x) = 0 \).
However, we can extend this function to the continuous function \( F: \mathbb{C} \to \mathbb{C} \), where
\( F(z) = z^2 + 1 \), and now the equation \( F(z) = 0 \) has solutions (notice that by extending
the map we change the nature of the map dramatically: the map \( F \) is surjective
wheres \( f \) was not).

So, what happens in general? Can we extend any continuous function \( f: \mathbb{R} \to \mathbb{R} \) to a
continuous function \( F: \mathbb{C} \to \mathbb{C} \)? Here is a variation of this question. Let \( A \subseteq \mathbb{R} \) be a
subset, and let \( f: A \to \mathbb{R} \) be a continuous function. Can we extend \( f \) to a continuous
function \( F: \text{Cl}(A) \to \mathbb{R} \)? There are many more variations of this question, and the
idea behind all of them is the following: if the domain and codomain of the function are missing some nice property, can we extend the function to bigger domain and or codomain, and still keep the continuity property? Well, sometimes we can, sometimes it is not possible.

Also related to continuous functions is the following. Let \( A \subseteq \mathbb{R} \) be a subspace, and let \( C(A) \) be the set of continuous functions \( f: A \to \mathbb{R} \). We can give this set the supremum metric:

\[
d(f, g) = \sup\{|f(x) - g(x)| \mid x \in A\}.
\]

Suppose we have a sequence of functions \( \{f_n\} \subseteq C(A) \) satisfying that they get closed to each other when \( n \) grows larger and larger (formally we would say that it is a Cauchy sequence, but let’s not worry about technicalities right now). What properties should \( A \) satisfy so that such a sequence converge in \( C(A) \)? Maybe the way this question was posed is a bit confusing, but you already know of an example of such problem: Taylor approximation of functions.

These and many more questions relate to the theory of metric and topological spaces. We have only uncovered the tip of the iceberg!
6. Connected spaces

Now that we have defined continuous maps between metric spaces, we can start comparing different metric spaces, trying to find properties that will distinguish one space from another. This chapter deals with the first of such properties: connectedness.

What defines connectivity for a metric space? This question turns out to be a bit tricky, since intuition from the most obvious examples would suggest something in the lines of “a metric space \((X, d_X)\) is connected if every two elements of \(X\) can be linked through a path in \(X\)” (think for instance of \(\mathbb{R}^2\) with the Euclidean metric). But we know better than trusting intuition by now.

If we give it a bit more thinking, a connected metric space should be one that cannot be formed out of smaller, disjoint pieces (compare \((-4, 3) \subseteq \mathbb{R}^2\) with \((-4, 2) \cup (2, 3) \subseteq \mathbb{R}^2\), always with the Euclidean metric).

Let then \((X, d_X)\) be a metric space. Intuitively, the idea that any two points of \(X\) can be linked by a path in \(X\) implies this other idea of \(X\) not being made of smaller, disjoint pieces. However, is the converse true? we will see later in this chapter that the answer is negative, and so we are presented now with two ideas about how to define connectivity for metric spaces, one of them seemingly stronger than the other.

Let’s start by formalizing what seems to be the weaker idea above. It actually turns out to be the right idea to define what a connected metric space is. The other idea will be discussed at the end of this chapter.

**Definition 6.1.** A metric space \((X, d_X)\) is connected if it satisfies the following.

(C) If a subset \(A \subseteq X\) is both open and closed then either \(A = X\) or \(A = \emptyset\).

Let’s see some examples of connected and non-connected spaces.

**Example 6.2.** Let \(X\) be a set with the discrete metric. Then, \(X\) is connected if and only if \(X\) is the empty set or a singleton (if \(X\) contains more than one element, then each sub-singleton of \(X\) is open and closed).

**Example 6.3.** Consider \(\mathbb{R}\) with the Euclidean metric, and let \(\mathbb{Q}\) be the metric subspace of rational numbers. Then, \(\mathbb{Q}\) is not connected. Indeed, the subset \((-\infty, \pi) \cap \mathbb{Q}\) is open and is also the complement of an open subset, hence closed.

**Example 6.4.** Consider the set \(\mathbb{Z}\) with the \(p\)-adic metric. This space is not connected, since the subset \(A = \{n \cdot p \mid n \in \mathbb{Z}\} \subseteq \mathbb{Z}\) of integers divisible by \(p\) is both open and closed. Indeed (see the exercise list 2), we have

\[
A = B_{0,1} \quad A = \mathbb{Z} \setminus \left( \bigcup_{n=1}^{p-1} B_{n,1} \right).
\]
The equality on the left means that $A$ is open, while the equality on the right says that $A$ is the complement of an open subset, hence closed.

As we have seen already several times in this course, it is always useful to have alternative statements for the same definition, so let’s see some alternative ways of defining connectivity.

**Theorem 6.5.** Let $(X, d_X)$ be a metric space. Then, the following are equivalent.

1. $(X, d_X)$ is connected.
2. There does not exist open subsets $U, V \subseteq X$ such that $U \cup V = X$ and $U \cap V = \emptyset$.
3. There does not exist any continuous map $f : X \to \{0\} \cup \{1\}$ which is surjective.

**Proof.** Let’s check first that (1) is equivalent to (3). Suppose first that there is a surjective, continuous map $f : X \to \{0\} \cup \{1\}$, and let $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$. Since $f$ is surjective, both $A$ and $B$ are non-empty subsets of $X$. Furthermore, since $\{0\}$ (respectively $\{1\}$) is an open and closed subset of $\{0\} \cup \{1\}$ and $f$ is continuous, it follows that $A$ (respectively $B$) is an open and closed subset of $X$. Finally, since both $A$ and $B$ are non-empty, it follows that none of them is either the empty set or $X$, and thus $X$ is not connected.

Suppose now that $X$ is not connected, and let $U \subsetneq X$ be a non-empty, open and closed subset. Let also $V = U^c$, and notice that it is again a non-empty, open and closed subset of $X$. We can then define a map $f : X \to \{0\} \cup \{1\}$ by setting $f(U) = \{0\}$ and $f(V) = \{1\}$, and it is clear that $f$ is surjective and continuous.

Finally let’s prove that (2) is equivalent to (3). If there exists a map $f$ as in (3), then $U = f^{-1}(\{0\})$ and $V = f^{-1}(\{1\})$ are open subsets of $X$ which clearly satisfy the conditions of (2). Conversely, if $U, V \subseteq X$ satisfy the conditions of (2) then we just have to define $f$ by $f(x) = 0$ if $x \in U$ and $f(x) = 1$ if $x \in V$. □

**Theorem 6.6.** Let $f : (X, d_X) \to (Y, d_Y)$ be a continuous map of metric spaces, and suppose $X$ is connected. Then, $f(X) \subseteq Y$ is a connected subspace of $Y$.

**Proof.** This is immediate using statement (2) in Theorem 6.5. □

**Corollary 6.7.** Let $h : (X, d_X) \to (Y, d_Y)$ be a homeomorphism of metric spaces. Then, $X$ is connected if and only if $Y$ is connected.

So, which spaces do we know to be connected so far? Looking at the examples above in this chapter, the only connected spaces mentioned are the empty set and a singleton, not too much to work with. In fact, most of the examples were of non-connected spaces! The first non-trivial examples of connected spaces that we can produce need a bit of hard work, but after that everything becomes easier.
Theorem 6.8. The set $\mathbb{R}$ with the Euclidean metric is connected.

Proof. Suppose $\mathbb{R} = A \cup B$, where $A, B$ are non-empty open subsets such that $A \cap B = \emptyset$. Then, $A$ and $B$ are both closed (they are the complement of each other), and thus

$$\partial(A) = \text{Cl}(A) \cap \text{Cl} (\mathbb{R} \setminus A) = \text{Cl}(A) \cap \text{Cl}(B) = A \cap B = \emptyset,$$

and similarly for $B$.

On the other hand, we will now show that if $U \subseteq \mathbb{R}$ is non-empty, then $\partial(U) \neq \emptyset$, hence contradicting the above. Let then $a \in \mathbb{R} \setminus U$. If $U \cap (a, \infty) = \emptyset = U \cap (-\infty, a)$, then $U = \{a\}$, and hence $U = \partial(U) \neq \emptyset$. Suppose then that $U \cap (a, \infty) \neq \emptyset$ (the other possible case is similar), and let $b = \inf\{x \in U \cap (a, \infty)\}$. Then we have

$$\inf\{d(b, u) \mid u \in U\} = 0 \quad \text{and} \quad \inf\{d(b, v) \mid v \in \mathbb{R} \setminus U\} = 0,$$

and this implies that $b \in \partial(U)$.

Now it is easy to deduce the following.

Example 6.9. For all $a < b \in \mathbb{R}$, the open interval $(a, b) \subseteq \mathbb{R}$ is connected. Indeed, $(a, b)$ is clearly homeomorphic to $(-\pi/2, \pi/2)$ (give the explicit homeomorphism!), and the map

$$\tan: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$$

is a homeomorphism.

So open intervals in $\mathbb{R}$ are connected. What about closed intervals? Their connectivity is a consequence of the following result.

Proposition 6.10. Let $(X, d_X)$ be a metric space and let $U, K \subseteq X$ be subsets. Suppose in addition that $U$ is connected and $U \subseteq K \subseteq \text{Cl}(U)$. Then $K$ is connected.

Proof. Let $f: K \rightarrow \{0\} \cup \{1\}$ be a continuous function. Since $U$ is connected and the restriction of $f$ to $U$ is a continuous function, then either $f(U) = \{0\}$ or $f(U) = \{1\}$. Suppose for simplicity that $f(U) = \{0\}$.

Now, for each $k \in K$, we have $k \in \text{Cl}(U)$. Let then $\{x_n\} \subseteq U$ be a sequence converging to $k$. Since $f$ is continuous, it follows then that the sequence $\{f(x_n)\}$ converges to $f(k)$. On the other hand, $f(x_n) = 0$ for all $n$, and thus $f(k) = 0$. This prove that $f(K) = 0$, and hence $K$ is connected by Theorem 6.5.

Theorem 6.11. Consider $\mathbb{R}$ with the Euclidean metric, and let $a < b \in \mathbb{R}$. Then, $[a, b] \subseteq \mathbb{R}$ is connected.

Proof. This follows immediately from Proposition 6.10 since $[a, b]$ is the closure of $(a, b)$, and the last is connected.
Once we know that \( \mathbb{R} \) is connected, the next step should be to check that \( \mathbb{R}^n \) is connected for all \( n \geq 2 \), but moving from dimension 1 to dimension higher or equal than 2 is a tricky step.

**Lemma 6.12.** Let \((X, d_X)\) be a metric space, and let \( \{A_\gamma\}_{\gamma \in \Gamma} \) be a collection of connected subsets of \( X \) such that

\[ A_\gamma \cap A_\omega \neq \emptyset \]

for all \( \gamma, \omega \in \Gamma \). Then, \( A = \bigcup_{\gamma \in \Gamma} A_\gamma \) is a connected subset of \( X \).

*Proof.* Let \( f : A \to \{0\} \cup \{1\} \) be a continuous function. Since \( A_\gamma \) is connected for all \( \gamma \in \Gamma \), we have

\[ f(A_\gamma) = \varepsilon_\gamma, \]

where \( \varepsilon_\gamma \) is either 0 or 1. However, for any two \( \gamma, \omega \in \Gamma \), we have \( A_\gamma \cap A_\omega \neq \emptyset \), and this implies that \( \varepsilon_\gamma = \varepsilon_\omega \). Hence, the map \( f \) has to be constant on \( A \): it cannot be surjective, and thus \( A \) is connected. \( \square \)

**Theorem 6.13.** Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces, and let \((X \times Y, d)\) be the product metric space. Then, \( X \times Y \) is connected if and only if \( X \) and \( Y \) are connected.

*Proof.* If \( X \times Y \) is connected, then both \( X \) and \( Y \) are connected, since the projection functions \( \pi_X : X \times Y \to X \) and \( \pi_Y : X \times Y \to Y \) are continuous by Exercise 4.21.

Suppose now that \( X \) and \( Y \) are connected. Fix then some element \( a \in X \) and let \( \Gamma = Y \) (as sets). Consider also the sets

- \( B = \{a\} \times Y \subseteq X \times Y \); and
- \( C_y = X \times \{y\} \subseteq X \times Y \), where \( y \in Y \).

Clearly, all the above subsets are connected subspaces of \( X \times Y \).

For each \( y \in Y \), we have \( C_y \cap B = \{(a, y)\} \neq \emptyset \), and hence the subset \( U_y = B \cup C_y \subseteq X \times Y \) is connected (for all \( y \in Y \)) by Lemma 6.12. Now, consider the collection \( \{U_y\}_{y \in Y} \). For any \( y, y' \in Y \), we have \( U_y \cap U_{y'} \neq \emptyset \), and hence

\[ \bigcup_{y \in Y} U_y = X \times Y \]

is connected by Lemma 6.12. \( \square \)

**Corollary 6.14.** For all \( n \geq 1 \), \( \mathbb{R}^n \) with the Euclidean metric is connected.

Let’s see now a less trivial example of connected space.

**Lemma 6.15.** Let \( f : (X, d_X) \to (Y, d_Y) \) be a continuous function, and suppose that \( X \) is connected. Then, the graph of \( f \), \( G_f = \{(x, y) \in X \times Y \mid y = f(x)\} \), is a connected subspace of \( X \times Y \).

*Proof.* By Exercise 4.25, \( G_f \) is homeomorphic to \( X \), and thus is connected. \( \square \)
Example 6.16. This example is known as the topologist’s sine curve. Let
\[ \Omega = \{(x, \sin(1/x)) \in \mathbb{R}^2 \mid x > 0\} \cup \{(0, y) \in \mathbb{R}^2 \mid y \in [-1, 1]\} \subseteq \mathbb{R}^2. \]
The following is a sketch of \( \Omega \).

To show that this space is connected, let
\[ \Gamma = \{(x, \sin(1/x)) \in \mathbb{R}^2 \mid x > 0\}. \]
First note that the subset \( \Gamma \) is connected because it is the graph of the continuous function \( f: (0, \infty) \to \mathbb{R} \) defined by \( f(x) = \sin(1/x) \), and \( (0, \infty) \) is connected. Finally, we have already seen in the tutorials that \( \Omega = \text{Cl}(\Gamma) \), and hence by Proposition 6.10 \( \Omega \) is connected. We will come back to this example in the next section.

6.1. Path-connected spaces. Which other spaces do you know to be connected? Maybe a circle or a disk in \( \mathbb{R}^2 \) (always with the Euclidean metric)? Regular polygons? Spheres? But how to prove that these spaces are connected? By applying Definition 6.1 or the alternative statements in Theorem 6.5, proving that any of these spaces is connected can become a titanic task!

Let’s try something different then. At the beginning of this chapter it was suggested to define connectivity of metric spaces in terms of linking any two points by a path. As mentioned, this is not the right way to define connectivity, but it is really helpful.

Definition 6.17. Let \( (X, d_X) \) be a metric space, and let \( x, y \in X \). A path in \( X \) from \( x \) to \( y \) is a continuous map \( \omega: [0, 1] \to X \) with \( \omega(0) = x \) and \( \omega(1) = y \).

A metric space \( (X, d_X) \) is path-connected if, for any two elements \( x, y \in X \) there is a path from \( x \) to \( y \).

Theorem 6.18. Let \( (X, d_X) \) be a path-connected space. Then \( (X, d_X) \) is connected.

Proof. Suppose that \( X \) is not connected, and let \( f: X \to \{0\} \cup \{1\} \) be a continuous, surjective map. Let also \( x \in f^{-1}(\{0\}) \) and \( y \in f^{-1}(\{1\}) \). Since \( X \) is path-connected,
there is some path $\omega : [0, 1] \to X$ with $\omega(0) = x$ and $\omega(1) = y$. However, the composition $f \circ \omega : [0, 1] \to \{0\} \cup \{1\}$ is both continuous (it is a composition of continuous functions) and is surjective, and this is impossible since $[0, 1]$ is connected by Theorem 6.11.

\[ \square \]

**Example 6.19.** For all $n \geq 1$, $\mathbb{R}^n$ is path-connected.

**Example 6.20.** A circle in $\mathbb{R}^2$ is the image of a path $\omega : [0, 1] \to \mathbb{R}$ which satisfies the extra condition $\omega(0) = \omega(1)$. Thus, a circle is connected.

**Exercise 6.21.** Let $(X, d_X)$ be a path-connected space, and let $x, y, z \in X$. Let also $\omega, \gamma : [0, 1] \to X$ be paths from $x$ to $y$ and from $y$ to $z$ respectively. How can you define a path from $x$ to $z$ using $\omega$ and $\gamma$?

**Corollary 6.22.** A cylinder is not homeomorphic to a Moebius band.

**Proof.** Let $M$ be a Moebius band and $C$ be a cylinder, both subsets of $\mathbb{R}^3$. Let also $\partial(M)$ and $\partial(C)$ be the boundaries of $M$ and $C$ are subsets of $\mathbb{R}^3$. Clearly, $\partial(C)$ is not connected, while $\partial(M)$ is path-connected (hence connected). Hence, no map $h : C \to M$ can be a homeomorphism.

It turns out that not all connected spaces are path-connected, and we have already met the first example of this behavior recently.

**Example 6.23.** Let $\Omega \subseteq \mathbb{R}^2$ be the subset in Example 6.16, the topologist’s sine curve, and recall that $\Omega$ is connected. Let’s prove now that $\Omega$ is not path-connected. Let then $A = \{(0, y) \in \mathbb{R}^2 \mid y \in [-1, 1]\}$ and $\Gamma = \{(x, \sin(1/x)) \in \mathbb{R}^2 \mid x > 0\}$.

If we choose elements $a \in A$ and $b \in \Gamma$, then there is no path in $\Omega$ linking $a$ to $b$! To prove this, suppose otherwise and let $\omega : [0, 1] \to \Omega$ be a path with $\omega(0) = (0, 0)$ and $\omega(1) = (\frac{\pi}{2}, 0)$. Let also $F : [0, 1] \to \mathbb{R}$ be the continuous map

$$F : [0, 1] \xrightarrow{\omega} \Omega \xrightarrow{\pi_1} \mathbb{R},$$

where $\pi_1(x, y) = x$. Since $F(0) = 0$ and $F(1) = \frac{1}{\pi}$, we can use the Intermediate Value Theorem to find some $t_1 \in (0, 1)$ such that $F(t_1) = \frac{2}{3\pi}$. Using again the Intermediate Value Theorem we can then find some $t_2 \in (0, t_1)$ such that $F(t_2) = \frac{4}{5\pi}$. Iterating the process we can find a sequence $\{t_n\}_{n \geq 1}$, such that

- $t_{n+1} \in (0, t_n)$; and
- $F(t_n) = \frac{2}{(2n+1)\pi}$.

Since the sequence $\{t_n\} \subseteq [0, 1]$ is monotonic decreasing and bounded below, it is convergent: there is some $t \in [0, 1]$ such that $\lim t_n = t$. Finally, since $\omega$ is continuous the sequence $\{\omega(t_n)\}$ converges to $\omega(t)$. On the other hand, $\omega(t_n) = (\frac{2}{(2n+1)\pi}, (-1)^n)$, and this sequence is not convergent, hence a contradiction.
6.2. **Connected components.** Connectivity constitutes then the first of a series of properties that we can use to characterize and distinguish metric spaces. Let’s elaborate this idea a little bit more: a connected space is one which is not made of smaller, disjoint pieces, but what happens in general? That is, given a metric space \((X, d_X)\), how many pieces are needed to “construct” \(X\)?

**Definition 6.24.** Let \((X, d_X)\) be a metric space. A **connected component** of \(X\) is a connected subset \(U \subseteq X\) such that

- if \(U \subseteq K \subseteq X\), then \(K\) is not connected.

The collection of connected components of \(X\) is denoted by \(\pi_0(X)\).

Alternatively, the set \(\pi_0(X)\) can be constructed as follows. Define a relation on the set \(X\): \(x \sim y\) if there is some connected subset \(U \subseteq X\) such that \(x, y \in U\). This is clearly an equivalence relation, and then we can define \(\pi_0(X)\) as the set of equivalence classes.

**Proposition 6.25.** Every continuous map \(f : (X, d_X) \to (Y, d_Y)\) induces a map of sets

\[ f_* : \pi_0(X) \to \pi_0(Y). \]

**Proof.** Let \(U \subseteq X\) be a connected component of \(X\), and let \(x \in U\). Let also \(V \subseteq Y\) be a connected component which contains \(f(x)\) (each \(y \in Y\) has to be contained in such a subset). We can define then \(f_*(U) = V\).

To check that this map is well-defined, let \(z \in U\) be any other element, and let \(w = f(z) \in Y\). We have to prove that \(f(z) \in V\). Suppose otherwise: \(f(z) \notin V\). Then \(V \not\subseteq W = V \cup \{f(z)\}\), and thus \(W\) is not connected. But this is impossible, since \(f(U)\) has to be connected. \(\square\)

**Corollary 6.26.** Let \(f : (X, d_X) \to (Y, d_Y)\) be a homeomorphism of metric spaces. Then \(f\) defines a bijection of sets \(f_* : \pi_0(X) \to \pi_0(Y)\).

In other words, if \(X\) and \(Y\) are homeomorphic, they have to be made out of the same number of pieces.
7. Compact spaces

We will now study one of the most important notions of this course, and (not surprisingly) one of the most difficult notions too: compactness of metric spaces. The idea is the following. If $X$ is a finite set (with the discrete metric) then studying (continuous) maps out of $X$ is reasonably easy, we just have to check what happens with the function on every element of $X$. But we want to consider other maps, not only maps out of finite sets!

We can think of **compactness** as some sort of generalization of **finiteness**, in the sense that maps out of a compact space behave nicely, as maps out of a finite set (this is, of course, rather rough, a proper justification will come soon in this chapter).

Before we define compactness for metric spaces, let’s review a few facts about metric spaces, subspaces and open subsets which will need in this chapter. Let then $(X, d_X)$ be a metric space: the metric $d_X$ is a function $d_X : X \times X \to [0, \infty)$ satisfying a certain list of conditions. Thus, given a subset $A \subseteq X$, we can automatically consider $A$ as a metric subspace of $X$, just by restricting the above function to $A \times A \subseteq X \times X$: the resulting function $d_A : A \times A \to [0, \infty)$ is a metric on $A$ (all this should be clear by now, but it is worth recalling).

Let then $(X, d_X)$ be a metric space and let $(A, d_A)$ be a metric subspace as above. Which is the relation (if any) between the set of open subsets of $A$ and the set of open subsets of $X$? It is as follows: for any open subset $U$ of $X$, the subset $V = U \cap A$ is an open subset of $A$ (check it!), and conversely, any open subset $V$ of $A$ is of the form $V = U \cap A$ for some open subset $U$ of $X$.

**Exercise 7.1.** Let $(X, d_X)$ be a metric space, and let $A \subseteq X$. Prove that $x \in \text{Cl}(A)$ if and only if $\inf \{d_X(x,a) \mid a \in A\} = 0$.

**Definition 7.2.** Let $(X, d_X)$ be a metric space and let $A \subseteq X$ be any subset. An **open cover** of $A$ in $X$ is a collection $\{U_\gamma\}_{\gamma \in \Gamma}$ of open subsets of $X$ such that

$$A \subseteq \bigcup_{\gamma \in \Gamma} U_\gamma.$$ 

A subcover of $\{U_\gamma\}_{\gamma \in \Gamma}$ is a subcollection $\{V_\omega\}_{\omega \in \Omega} \subseteq \{U_\gamma\}_{\gamma \in \Gamma}$ such that $A \subseteq \bigcup_{\Omega} V_\omega$.

Thus, let $(X, d_X)$ be a metric space and let $A \subseteq X$. Then, it is exactly the same to talk about open covers of $A$ in $X$, or open covers of $A$ in $(A, d_A)$: if $\{U_\gamma\}_{\gamma \in \Gamma}$ is an open cover of $A$, then the collection

$$\{W_\gamma = U_\gamma \cap A\}_{\gamma \in \Gamma}$$

is a collection of open subsets of $(A, d_A)$, and $A = \bigcup_{\Gamma} W_\gamma$. 


Example 7.3. Consider the family of intervals $U_n = (n, n + 2) \subseteq \mathbb{R}$ (Euclidean metric), for $n \in \mathbb{Z}$. The collection $\{U_n\}_{n \in \mathbb{Z}}$ is an open cover of $\mathbb{R}$. Notice also that, if for some $n \in \mathbb{Z}$ we remove $U_n$ from the collection, then it stops being an open cover of $\mathbb{R}$. In other words, this open cover does not contain any proper subcover.

And we are now ready to define compactness for metric spaces. This definition is rather complicated, so take your time to think about it.

Definition 7.4. Let $(X, d_X)$ be a metric space and let $A \subseteq X$. The metric subspace $(A, d_A)$ is

1. **bounded** if there exist $x \in X$ and $\varepsilon > 0$ such that $A \subseteq B_{x, \varepsilon}$;
2. **compact** if every open cover of $A$ in $X$ contains a finite subcover.

Remark 7.5. Let $(X, d_X)$ be a metric space and $A \subseteq X$. Let also $\{U_\gamma\}_{\gamma \in \Gamma}$ be an open cover of $A$ in $X$. A finite subcover is then a finite subcollection $\{V_\omega\}_{\omega \in \Omega} \subseteq \{U_\gamma\}_{\gamma \in \Gamma}$ which is also an open cover of $A$. This is already a hint of the idea that compactness generalizes finiteness of metric spaces.

**WARNING!** As a note of caution, be extremely careful with the above definition. Specially with the bit *every open cover*... It is not enough to check that an open cover contains a finite subcover, we have to check ALL possible open covers! Indeed, notice that, for any metric space $(X, d_X)$, the collection $\{X\}$ is a finite open cover of $X$, but this does not make $X$ compact!

The best way to understand this notion is by looking at some examples. Let’s start with example of spaces which are not compact.

Example 7.6. Consider $\mathbb{R}$ with the Euclidean metric. Then, the open cover in Example 7.3 does not contain any finite subcover, and hence $\mathbb{R}$ is not compact.

Example 7.7. Consider $\mathbb{R}$ with the Euclidean metric, and let $(0, 1) \subseteq \mathbb{R}$ be the open interval. Then $(0, 1)$ is not compact. Indeed, consider the open cover

$$\left\{ U_n = \left( \frac{1}{n}, 1 \right) \right\}_{n \in \mathbb{N}}$$

It is easy to see that this open cover does not contain any finite subcover of $(0, 1)$. This is actually not surprising since we already know that $(0, 1)$ is homeomorphic to $\mathbb{R}$.

And what about some example of compact space?

Example 7.8. Let $X$ be a finite set with the discrete metric. Then it is clear that $X$ is compact: since $X$ contains a finite number of elements, there is a finite number of open subsets of $X$, and hence any open cover of $X$ is already finite.
Fine, but what about a nontrivial example of a compact space? Let’s start with something (relatively) easy: the compact subsets of \( \mathbb{R} \) (with the Euclidean metric).

**Theorem 7.9 (Heine-Borel Theorem for \( \mathbb{R} \)).** Consider \( \mathbb{R} \) with the Euclidean metric, and let \( A \subseteq \mathbb{R} \). Then, \( A \) is compact if and only if it is closed and bounded.

**Proof.** Suppose first that \( A \) is compact. To see that it is bounded, consider the following open cover of \( A \):

\[
\{ U_n = B_{0,n} \mid n \in \mathbb{N} \}.
\]

Since it is an open cover of \( A \) and \( A \) is compact, there exists some finite subcover, namely \( \{ U_{n_1}, \ldots, U_{n_r} \} \). Let \( N = \max\{n_1, \ldots, n_r\} \). Clearly, \( U_{n_1}, \ldots, U_{n_r} \subseteq U_N \), and hence \( A \subseteq U_N = B_{1,N} \), i.e. \( A \) is bounded.

Let’s see also that \( A \) is closed. Suppose otherwise, and let \( x \in \text{Cl}(A) \setminus A \). For each \( n \in \mathbb{N} \), let

\[
U_n = \left( \text{Cl}(B_{x,\frac{1}{n}}) \right)^c = \mathbb{R} \setminus \text{Cl}(B_{x,\frac{1}{n}}).
\]

By definition, each \( U_n \) is the complement of a closed subset of \( \mathbb{R} \) and thus is open. Furthermore, we have

\[
\bigcup_{n \in \mathbb{N}} U_n = \bigcup_{n \in \mathbb{N}} \mathbb{R} \setminus \text{Cl}(B_{x,\frac{1}{n}}) = \mathbb{R} \setminus \bigcap_{n \in \mathbb{N}} \text{Cl}(B_{x,\frac{1}{n}}) = \mathbb{R} \setminus \{ x \} \supseteq A.
\]

In other words, \( \{ U_n \}_{n \in \mathbb{N}} \) is an open cover of \( A \) in \( \mathbb{R} \), and since \( A \) is compact, it contains a finite subcover, \( \{ U_{n_1}, \ldots, U_{n_s} \} \). Furthermore, by definition of the subsets \( U_n \), we have \( U_{n_1}, \ldots, U_{n_s} \subseteq U_N \), where \( N = \max\{n_1, \ldots, n_s\} \), and also \( A \subseteq U_N \). But then we have

\[
A \cap B_{x, \frac{1}{N}} = \emptyset,
\]

which contradicts the hypothesis that \( x \in \text{Cl}(A) \). Hence \( A = \text{Cl}(A) \).

Let’s prove now the converse. Assume then that \( A \) is closed and bounded, and let \( \{ U_\gamma \}_{\gamma \in \Gamma} \) be an open cover of \( A \). Since \( A \) is closed and bounded, there exist \( m \) and \( M \in A \) such that

\[
m \leq a \leq M \quad \forall a \in A.
\]

Now, for each \( x \in \mathbb{R} \), define the following sets:

- \( W_x = (-\infty, x] \); and
- \( A_x = A \cap W_x \).

Define also \( B = \{ x \in \mathbb{R} \mid A_x \text{ is covered by a finite subcover of } \{ U_\gamma \} \} \), and note that \( B \) is non-empty, since \( m \in B \). Indeed, \( A_m = A \cap W_m = \{ m \} \) by definition of \( m \), and it is clear that \( A_m \) is covered by a finite subcover of \( \{ U_\gamma \} \).

Notice also that, to finish the proof, it is enough to show that \( B \) is not bounded above: if this is the case, then \( M \leq y \) for some \( y \in B \), and then by definition of \( B \) we have that \( A = A_M = A_y \) is covered by a finite subcover of \( \{ U_\gamma \} \).
We will prove that $B$ is not bounded above by contradiction. Suppose then that $u = \sup B$ exists: $y \leq u$ for all $y \in B$. If $u \notin A$, then there exists some $\varepsilon > 0$ such that $B_{u,\varepsilon} \cap A = \emptyset$ (this is because $A$ is closed). This implies that

$$A_{u-\varepsilon} = A_{u+\varepsilon}.$$  

But notice that $u-\varepsilon < u$, and hence $u-\varepsilon \in B$. The above implies then that $u+\varepsilon \in B$, and this contradicts the hypothesis that $u = \sup B$.

Thus we have to assume that $u \in A$. In this case, since $\{U_\gamma\}$ is an open cover of $A$, there is some $U_\omega \in \{U_\gamma\}$ such that $u \in U_\omega$. Furthermore, since $U_\omega$ is open, we can find some $\delta > 0$ such that

$$[u - \delta, u + \delta] \subseteq U_\omega.$$  

Now, since $u - \delta < u$, the set $A_{u-\delta}$ is covered by a finite subcover of $\{U_\gamma\}$, namely $\{U_{\gamma_1}, \ldots, U_{\gamma_m}\}$, and then $\{U_\omega, U_{\gamma_1}, \ldots, U_{\gamma_m}\}$ is a finite subcover of $A_{u+\delta}$. Again, this implies that $u + \delta \in B$, which contradicts the hypothesis that $u = \sup B$. Hence, $B$ is not bounded above, and $A$ is compact. 

We can use this theorem to deduce that some other subsets of $\mathbb{R}$ are compact. The following example is not intuitive at all.

**Example 7.10.** Let $\mathcal{K} \subseteq \mathbb{R}$ be the Cantor set (see Exercise 6 in List 3): start from $I_0 = [0, 1] \subseteq \mathbb{R}$, then form $I_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ by removing the middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$ of $I_0$, and keep iterating the process. We end up with a sequence $\{I_n\}_{n \in \mathbb{N}}$ of subsets of $\mathbb{R}$, and

$$\mathcal{K} = \bigcap_{n=0}^{\infty} I_n.$$  

Since each $I_n$ is closed and $\mathcal{K}$ is an (infinite) intersection of closed subsets, we know that $\mathcal{K}$ is closed. Also, it is clearly bounded, since $\mathcal{K} \subseteq I_0$, and hence by the Heine-Borel it is compact.

A version of the Heine-Borel Theorem is also true when we replace $\mathbb{R}$ by $\mathbb{R}^n$, for any $n \geq 1$. However, the proof for Theorem 7.9 does not work in general (why?).

Thus, before attempting to generalize Theorem 7.9, let’s study the behavior of compact subsets under different situations: what about subsets of compact spaces? And about compact spaces and continuous maps? Finally, last but not least, what about the product of two compact spaces? Each of these questions is answered below and will be needed to prove the general version of the Heine-Borel Theorem.

If you check carefully the proof of the Heine-Borel Theorem for $\mathbb{R}$, the proof that a compact subset of $\mathbb{R}$ is closed and bounded does not pose any complication. In fact, this turns out to be a general statement.

**Proposition 7.11.** Let $(X, d_X)$ be a metric space, and let $A \subseteq X$ be a compact subspace. Then, $A$ is closed and bounded.
Proof. The same proof as in Theorem 7.9 (with minor modifications) works here, but let’s reproduce it anyway. First let’s check that $A$ is bounded. Fix some $x \in X$ and let $U_n = B_x, n$ for each $n \in \mathbb{N}$. The collection $\{U_n\}_{n \in \mathbb{N}}$ is thus an open cover of $A$ in $X$, and since $A$ is compact there exists some finite subcover, $\{U_{n_1}, \ldots, U_{n_r}\}$. Let then $N = \max\{n_1, \ldots, n_r\}$: we have $A \subseteq U_N = B_x, N$ and hence $A$ is bounded.

To see that $A$ is also closed, suppose otherwise and let $x \in \text{Cl}(A) \setminus A$. For each $n \in \mathbb{N}$ define then

$$U_n = (\text{Cl}(B_x, \frac{1}{n}))^c = X \setminus \text{Cl}(B_x, \frac{1}{n}).$$

Then, $U_n$ is open for all $n$ and we have

$$\bigcup_{n \in \mathbb{N}} U_n = \bigcup_{n \in \mathbb{N}} X \setminus \text{Cl}(B_x, \frac{1}{n}) = X \setminus \bigcap_{n \in \mathbb{N}} \text{Cl}(B_x, \frac{1}{n}) = X \setminus \{x\} \subseteq A.$$

Thus, $\{U_n\}_{n \in \mathbb{N}}$ is an open cover of $A$, and since $A$ is compact there is some finite subcover, $\{U_{n_1}, \ldots, U_{n_r}\}$. Furthermore, by setting $N = \max\{n_1, \ldots, n_s\}$, we have $A \subseteq U_N$, which implies

$$A \cap B_{x, \frac{1}{N}} = \emptyset,$$

contradicting the hypothesis that $x \in \text{Cl}(A)$. Thus, $A = \text{Cl}(A)$. \hfill $\Box$

Lemma 7.12. Let $(X, d_X)$ be a metric space and let $A \subseteq X$ be a compact subset. Then, each closed subset $K \subseteq A$ is compact.

Proof. Let $\{U_\gamma\}_{\gamma \in \Gamma}$ be an open cover of $K$ in $X$. Since $K$ is closed, its complement $V = X \setminus K$ is open, and the collection $\{U_\gamma\}_{\gamma \in \Gamma} \cup V$ is an open cover of $A$. Thus, since $A$ is compact, there exists some finite subcover, $\{W_1, \ldots, W_n\}$.

If $\{W_1, \ldots, W_n\} \subseteq \{U_\gamma\}_{\gamma \in \Gamma}$ we are finished, since this is already a finite subcover of $K$. If the subset $V$ is one of the elements of $\{W_1, \ldots, W_n\}$, then by removing it from the collection we obtain a finite subcover of $K$. \hfill $\Box$

Let’s see now how compact subsets behave when we apply to them continuous functions. The first property was to be expected: the image of a compact subset through a continuous function is again compact.

Proposition 7.13. Let $f : (X, d_X) \to (Y, d_Y)$ be a continuous map, and let $A \subseteq X$ be a compact subset. Then, $f(A) \subseteq Y$ is compact.

Proof. Let $\{U_\gamma\}_{\gamma \in \Gamma}$ be an open cover of $f(A)$ in $Y$. Since $f$ is a continuous map, the collection $\{f^{-1}(U_\gamma)\}_{\gamma \in \Gamma}$ is then an open cover of $A$. Now, since $A$ is compact we know that there is some finite subcover $\{f^{-1}(U_{\gamma_1}), \ldots, f^{-1}(U_{\gamma_r})\}$, and thus $\{U_{\gamma_1}, \ldots, U_{\gamma_r}\}$ is a finite subcover of $f(A)$. \hfill $\Box$

Let $f : [0, 1] \to \mathbb{R}$ be a continuous map. It is a well-known fact that the image of this map is then bounded in $\mathbb{R}$. But now that we know a bit about compact spaces, we can analyze what is happening here: $[0, 1]$ is a compact space, and thus the image
of $f$, $\operatorname{Im}(f)$, is a compact subset of $\mathbb{R}$. Furthermore, since $[0,1]$ is connected, so is $\operatorname{Im}(f)$. Hence, $\operatorname{Im}(f) = [a,b]$ for some $a < b \in \mathbb{R}$! It is clear now that $\operatorname{Im}(f)$ is bounded above and below.

**Corollary 7.14.** Let $f : (X,d_X) \to (Y,d_Y)$ be a continuous map, and let $A \subseteq X$ be a compact subset. Then, $f(A)$ is bounded in $Y$.

**Corollary 7.15.** Let $f : (X,d_X) \to (Y,d_Y)$ be a homeomorphism. Then, $X$ is compact if and only if $Y$ is compact.

**Example 7.16.** Recall from the tutorials that the Cantor set, $\mathcal{K}$, in Example 7.10, has the cardinality of the set $\mathbb{R}$ of real numbers. But now we know that $\mathcal{K}$ is compact, while $\mathbb{R}$ is not. Thus, these sets are not homeomorphic, despite having the same cardinality!

The last property of compact subsets that we need in order to prove the Heine-Borel Theorem for $\mathbb{R}^n$ turns out to be also the most complicated of all.

**Theorem 7.17.** Let $(X,d_X)$ and $(Y,d_Y)$ be metric spaces, and let $A \subseteq X$, $B \subseteq Y$ be compact subsets. Then, $A \times B$ is a compact subset of $X \times Y$.

**Proof.** Let $\{W_\gamma\}_{\gamma \in \Gamma}$ be an open cover of $A \times B$. In order to make this proof easier to follow, let’s divide it in two main parts.

- We can assume that, for all $\gamma \in \Gamma$, $W_\gamma = U_\gamma \times V_\gamma$, where $U_\gamma \subseteq X$ and $V_\gamma \subseteq Y$ are open subsets.

Recall that, with the product metric $d$ on $X \times Y$, we have

$$B^d_{(x,y),\varepsilon} = B^{d_X}_{x,\varepsilon} \times B^{d_Y}_{y,\varepsilon}$$

for all $(x,y) \in X \times Y$ and all $\varepsilon > 0$.

Now, fix some $\gamma \in \Gamma$. Since $W_\gamma \subseteq X \times Y$ is open, for each $(x,y) \in W_\gamma$ there exists some $\varepsilon > 0$ (which depends on $(x,y)$) such that $B^d_{(x,y),\varepsilon} \subseteq W_\gamma$, and

$$W_\gamma = \bigcup_{(x,y) \in W_\gamma} B^d_{(x,y),\varepsilon}.$$

We can then consider the following open cover of $A \times B$

$$\{B^d_{(x,y),\varepsilon} \mid (x,y) \in W_\gamma\}_{\gamma \in \Gamma}$$

If this open cover contains a finite subcover then the original open cover will also contain a finite subcover (check it!).

- Let $\{W_\gamma\}_{\gamma \in \Gamma}$ be an open cover of $A \times B$, with $W_\gamma = U_\gamma \times V_\gamma$ for all $\gamma$. Then, this open cover contains a finite subcover.
For each \( a \in A \), consider the subset \( \{a\} \times B \subseteq A \times B \). This subset is homeomorphic to \( B \), which is a compact subset of \( Y \) by hypothesis, and thus \( \{a\} \times B \) is compact too, by Corollary 7.15. Since \( \{W_\gamma\} \) also covers \( \{a\} \times B \), there exists some finite subcover \[ \Omega_a = \{W_{a,i_1}, \ldots, W_{a,i_a}\}. \]

The following picture illustrates the situation.

Recall that, for \( k = i_1, \ldots, i_a \), \( W_{a,k} = U_{a,k} \times V_{a,k} \) for some \( U_{a,k} \) and \( V_{a,k} \) open in \( A \) and \( B \) respectively. Define then
\[ U_a = U_{a,i_1} \cap U_{a,i_2} \cap \ldots \cap U_{a,i_a}. \]
This is a non-empty open subset of \( X \), and thus the collection \( \{U_a\}_{a \in A} \) is an open cover of \( A \). Since \( A \) is compact, there exist \( a_1, \ldots, a_m \in A \) such that
\[ \{U_{a_1}, \ldots, U_{a_m}\} \subseteq \{U_a\}_{a \in A} \]
is a finite subcover.

This finishes the proof, because the collection
\[ \Omega_{a_1} \cup \ldots \cup \Omega_{a_m} = \{W_{a_1,i_1}, \ldots, W_{a_1,i_{a_1}}\} \cup \ldots \cup \{W_{a_m,i_1}, \ldots, W_{a_m,i_{a_m}}\} \subseteq \{W_\gamma\}_{\gamma \in \Gamma} \]
is a finite subcover of \( A \times B \). \( \square \)

**Corollary 7.18.** Let \( (X, d_X) \) and \( (Y, d_Y) \) be metric spaces, and let \( A \subseteq X \), \( B \subseteq Y \) be subsets. Then, \( A \) and \( B \) are compact if and only if \( A \times B \) is a compact subset of \( X \times Y \).

We can finally prove the Heine-Borel Theorem in its general version.
Theorem 7.19 (Heine-Borel Theorem for $\mathbb{R}^n$). Consider $\mathbb{R}^n$ with the Euclidean metric, and let $A \subseteq \mathbb{R}^n$ be a subset. Then, $A$ is compact if and only if it is closed and bounded.

Proof. By Proposition 7.11, if $A$ is compact then it is closed and bounded. Suppose then that $A$ is closed and bounded. In particular, there is some interval $[m, M] \subseteq \mathbb{R}$ such that $A \subseteq I = [m, M] \times \ldots \times [m, M] \subseteq \mathbb{R}^n$.

Now, the subset $I \subseteq \mathbb{R}^n$ is compact by Theorem 7.17, since $I$ is the direct product of compact subsets of $\mathbb{R}$, and then by Lemma 7.12 $A$ is compact since it is a closed subset of $I$. □

We can use now the Heine-Borel Theorem to prove that a certain subset of $\mathbb{R}$ is compact.

Example 7.20. Consider $\mathbb{R}^2$ with the Euclidean metric. Then, every closed disk and every circle is compact. More generally, in the Euclidean space $\mathbb{R}^n$, the sphere $S^{n-1}$,

$$S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \ldots + x_n^2 = 1\}$$

is a compact space.

Example 7.21. The cylinder and the Moebius band are both compact subsets of $\mathbb{R}^3$. Thus compactness is not a good property to distinguish these two subsets. Fortunately connectivity has done the job for us already.

The Heine-Borel Theorem is really nice: it gives an easy-to-apply criterion to decide whether $A \subseteq \mathbb{R}^n$ is compact or not. It would be great if such a tool was available for all metric spaces, wouldn’t it? However, things are not that nice, and a counterexample arises immediately.

Example 7.22. Consider the set $\mathbb{Z}$ with the discrete metric. Then, $\mathbb{Z}$ is closed and bounded, but not compact. Indeed, $\mathbb{Z}$ is closed as a subset of itself, and it is bounded because $\mathbb{Z}$ is contained in the open ball of radius 2 around any element. However, every singleton $\{n\}$ in $\mathbb{Z}$ is an open subset, and hence the collection $\{\{n\}\}_{n \in \mathbb{Z}}$ is an open cover which does not contain any finite subcover.

Compactness is a rather strong property, as we already know: think for instance of the difference between continuous functions from $(0, 1)$ (which is not compact) to $\mathbb{R}$ and continuous functions from $[0, 1]$ (which is compact) to $\mathbb{R}$. In general, it is better to work with compact spaces, but as we have seen in this section it is difficult in general to show that a space is compact. The goal is then to find some criterion for compactness. This will be done in the next chapter.
7.1. **Continuity improved: uniform continuity.** In this short section we show how compactness affects to continuity of function.

**Definition 7.23.** A map \( f : (X, d_X) \rightarrow (Y, d_Y) \) between metric spaces is **uniformly continuous** if, for all \( \varepsilon > 0 \) there exists some \( \delta > 0 \) such that

\[
\text{if } d_X(x, y) < \delta \text{ then } d_Y(f(x), f(y)) < \varepsilon.
\]

Notice the difference with the definition of a continuous function (Definition 4.3): here, \( \delta \) may depend on \( \varepsilon \), but not on the elements \( x \) and \( y \). Thus, this is a stronger notion than usual continuity.

**Proposition 7.24.** Let \( f : (X, d_X) \rightarrow (Y, d_Y) \) be a continuous function of metric spaces, and suppose in addition that \( X \) is compact. Then \( f \) is uniformly continuous.

*Proof.* Let \( \varepsilon > 0 \). Since \( f \) is a continuous map, we know that for each \( x \in X \) there exists some \( \delta(x) > 0 \) such that

\[
\text{if } d_X(x, y) < 2\delta(x) \text{ then } d_Y(f(x), f(y)) < \frac{\varepsilon}{2}.
\]

Clearly, the collection \( \{B_{x,\delta(x)}\}_{x \in X} \) is an open cover of \( X \), and since \( X \) is compact we know that there exist \( x_1, \ldots, x_m \in X \) such that \( \{B_{x_1,\delta(x_1)}, \ldots, B_{x_m,\delta(x_m)}\} \) is a finite subcover of \( X \).

Let then \( \delta = \min\{\delta(x_1), \ldots, \delta(x_m)\} \), and let \( x, y \in X \) be such that \( d_X(x, y) < \delta \). Since the above is a finite subcover of \( X \), we have \( x \in B_{x_k,\delta(x_k)} \) for some \( x_k \) as above, and then by the triangle inequality

\[
d_X(x_i, y) \leq d_X(x_i, x) + d_X(x, y) < \delta(x_i) + \delta \leq 2\delta(x_i).
\]

By the continuity of \( f \) at the element \( x_i \), we have

\[
d_X(x_i, x) < \delta(x_i) < 2\delta(x_i) \implies d_Y(f(x_i), f(x)) < \frac{\varepsilon}{2}
\]

\[
d_X(x_i, y) < 2\delta(x_i) \implies d_Y(f(x_i), f(y)) < \frac{\varepsilon}{2}
\]

and hence using the triangle inequality and the symmetry property of \( d_Y \)

\[
\text{if } d_X(x, y) < \delta \text{ then } d_Y(f(x), f(y)) \leq d_Y(f(x_i), f(x)) + d_Y(f(x_i), f(y)) < \varepsilon
\]

and the proof is finished. \( \square \)
8. Complete spaces

The main purpose of this chapter is to give conditions for a space to be compact. Surprisingly enough, in this chapter sequences and their convergence become the main character. First of all, let’s recall the notion of Cauchy sequence. This notion has already appeared briefly in some tutorials and in the continuous assessment.

**Definition 8.1.** Let $(X, d_X)$ be a metric space. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in $X$ is a Cauchy sequence if, for all $\varepsilon > 0$, there exists some $M \in \mathbb{N}$ such that

$$d_X(x_n, x_m) < \varepsilon \text{ for all } n, m \geq M.$$ 

Recall from the tutorials that every convergent sequence is a Cauchy sequence, but the converse is not necessarily true (the $p$-adic metric on $\mathbb{Z}$ provides examples of this).

**Definition 8.2.** A metric space $(X, d_X)$ is complete if every Cauchy sequence is convergent.

Thus, in a complete space a sequence is convergent if and only if it is Cauchy.

**Example 8.3.** Let $X$ be a set with the discrete metric. Then $X$ is a complete space. Indeed, let $\{x_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence, and let $\varepsilon = \frac{1}{2}$. Then, there exists some $N \in \mathbb{N}$ such that, for all $n, m \geq N$,

$$d(x_n, x_m) < \frac{1}{2},$$

which implies that $d(x_n, x_m) = 0$. In other words, the sequence is constant for all $n \geq N$, and hence it is convergent. This is also an example of the fact that completeness does not imply compactness!

Before studying other examples, notice that the definition of Cauchy sequence only involves the terms of the sequence, whereas the notion of convergent sequence involves also the limit of the sequence (which we may not know a priori).

**Example 8.4.** Consider the sequence $\{x_n\}_{n \in \mathbb{N}}$, where $x_n = \sum_{k=1}^{n} \frac{1}{k^2}$. Let also $n, m \in \mathbb{N}$, and suppose $n \geq m$ for simplicity. Then,

$$|x_n - x_m| = x_n - x_m = \sum_{k=m+1}^{n} \frac{1}{k^2} \leq \sum_{k=m+1}^{n} \left(\frac{1}{k} - \frac{1}{k-1}\right) = \frac{1}{m} - \frac{1}{n} \leq \frac{1}{m} + \frac{1}{n}.$$

Now it is easy to check that $\{x_n\}$ is a Cauchy sequence. Actually, this sequence is also convergent (we will show soon that $\mathbb{R}$ is complete). What is the limit of this sequence?

The next result is probably known to you already, but it is nevertheless worth reviewing.
**Theorem 8.5.** The set \( \mathbb{R} \) with the Euclidean metric is complete.

**Proof.** Let \( \{x_n\}_{n \in \mathbb{N}} \) be a Cauchy sequence of real numbers. First, let’s check that the sequence must be bounded. By applying the Cauchy condition with \( \varepsilon = 1 \) it follows that there is some \( N \in \mathbb{N} \) such that, for all \( n, m \geq N \),

\[
|x_n - x_m| < 1.
\]

In particular, the above inequality is true for \( m = N \) and all \( n \geq N \). We have

\[
|x_n| - |x_N| \leq |x_n - x_N| < 1,
\]

and hence \( |x_n| < 1 + |x_N| \). Thus, it is clear that the sequence is bounded, since for all \( n \in \mathbb{N} \) we have

\[
|x_n| \leq \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, 1 + |x_N|\}.
\]

Next, out of \( \{x_n\}_{n \in \mathbb{N}} \) we construct a new sequence, \( \{y_n\}_{n \in \mathbb{N}} \), which is convergent and bounds \( \{x_n\} \) in a particular way. Let

\[
A_n = \{x_m \mid m \geq n\},
\]

and note that \( A_1 = \{x_n\}_{n \in \mathbb{N}} \), and \( A_{n+1} \subseteq A_n \) for all \( n \). Furthermore, since the set \( A_1 \) is bounded, so is \( A_n \) for each \( n \). We may consider then the supremum of \( A_n \):

\[
y_n = \sup(A_n).
\]

The sequence \( \{y_n\}_{n \in \mathbb{N}} \) is monotonic decreasing, since \( A_{n+1} \subseteq A_n \), and is bounded below because \( A_1 \) is bounded below and \( y_n \geq x_n \) for all \( n \). Thus, this new sequence is convergent. Let

\[
L = \lim_{n \to \infty} y_n.
\]

The proof is finished if we show that \( \{x_n\}_{n \in \mathbb{N}} \) also converges to \( L \). Given \( \varepsilon > 0 \), there exist

1. \( N_1 \in \mathbb{N} \) such that, for all \( n, m \geq N_1 \), \( |x_n - x_m| < \varepsilon \);
2. \( N_2 \in \mathbb{N} \) such that, for all \( k \geq N_2 \), \( |L - y_k| < \varepsilon \).

Let \( N = \max\{N_1, N_2\} \). Since \( y_N = \sup(A_N) \), there must exist some \( M \geq N \) such that

\[
x_M > y_N - \varepsilon \quad \text{and} \quad x_M \leq y_N.
\]

Thus, for all \( n \geq N \), we have

\[
|L - x_n| = |L - y_N + y_N - x_M + x_M - x_n| \leq |L - y_N| + |y_N - x_M| + |x_M - x_n| < 3\varepsilon
\]

which proves that \( L \) is the limit of \( \{x_n\}_{n \in \mathbb{N}} \). \( \square \)

We can use this example to study other examples.
Example 8.6. The set of rational numbers $\mathbb{Q}$ with the Euclidean metric is not complete. For instance, let
\[ x_n = \frac{\lfloor 10^n \cdot \pi \rfloor}{10^n}, \]
where $\lfloor y \rfloor$ denotes the greatest integer which is smaller or equal to $y$. In this case, $x_n$ keeps the first $n$ decimal terms of $\pi$ and ignores the rest. Clearly, $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence, but it is not convergent in $\mathbb{Q}$.

Example 8.7. The open interval $(0, 1) \subseteq \mathbb{R}$ with the Euclidean metric is not complete. For instance the sequence $\{x_n = \frac{1}{2^n}\}_{n \in \mathbb{N}}$ is Cauchy but not convergent.

The above example is actually rather important: recall from Example 6.9 that $(0, 1)$ is homeomorphic to $\mathbb{R}$. On the other hand, $\mathbb{R}$ is complete while $(0, 1)$ is not.

Corollary 8.8. Completeness is not preserved by homeomorphisms.

Following the usual program of previous chapters, we next study whether $\mathbb{R}^n$ is also complete, for $n \geq 2$.

Theorem 8.9. For all $n \geq 1$, the Euclidean space $\mathbb{R}^n$ is complete.

Proof. By Theorem 8.5 we can assume that $n \geq 2$. Let then $\{x_k\}_{k \in \mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}^n$, where
\[ x_k = (x_{1,k}, \ldots, x_{n,k}) \]
for all $k \in \mathbb{N}$. Now, the sequence $\{x_{i,k}\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ for each $i = 1, \ldots, n$ (check the details), and hence it is convergent. For each $i$ write
\[ L_i = \lim_{k \to \infty} x_{i,k}. \]
To finish the proof we have to check that $\{x_k\}_{k \in \mathbb{N}}$ converges to the point $L = (L_1, \ldots, L_n)$. Since each of the sequences $\{x_{i,k}\}_{k \in \mathbb{N}}$ is convergent, for each $\varepsilon > 0$ there exist $N_1, \ldots, N_n \in \mathbb{N}$ such that
\[ \text{if } k \geq N_i \text{ then } |L_i - x_{i,k}| < \varepsilon. \]
Let then $N = \max\{N_1, \ldots, N_n\}$. If $k \geq N$, then
\[ d(L, x_k) = \sqrt{\Sigma_{i=1}^n (L_i - x_{i,k})^2} < \sqrt{n} \cdot \varepsilon. \]
Hence, the sequence $\{x_k\}_{k \in \mathbb{N}}$ is convergent, and $\mathbb{R}^n$ is complete. \qed

The following, more general result will be proved in the tutorials (the proof above is a good guideline to prove the theorem below).

Theorem 8.10. Let $(X_1, d_1), \ldots, (X_n, d_n)$ be metric spaces. Then, $X_1 \times \ldots \times X_n$ (with the product metric) is complete if and only if $(X_i, d_i)$ is complete for all $i = 1, \ldots, n$. 

Do you know of other complete metric spaces? Well, in fact you do: remember Exercise 3 in the continuous assessment?

**Example 8.11.** Let \( C[0, 1] \) be the set of continuous functions from \([0, 1]\) to \( \mathbb{R} \), with the metric

\[
d(f, g) = \max\{|f(x) - g(x)| \mid x \in [0, 1]\}.
\]

Then, \( C[0, 1] \) is complete, as was proved in the continuous assessment.

However, the same set \( C[0, 1] \) with the following metric is not complete any more:

\[
d'(f, g) = \int_0^1 |f(x) - g(x)| \, dx.
\]

Indeed, consider the sequence of functions \( \{f_n\}_{n \in \mathbb{N}} \), where

\[
f_n(x) = \begin{cases} 
0, & x \in \left[0, \frac{1}{2} - \frac{1}{n}\right] \\
 n \cdot \left(x + \frac{1}{n} - \frac{1}{2}\right), & x \in \left[\frac{1}{2} - \frac{1}{n}, \frac{1}{2}\right] \\
1, & x \in \left[\frac{1}{2}, 1\right]
\end{cases}
\]

As an exercise, check that this sequence is indeed a Cauchy sequence with respect to the metric \( d' \) above. We have to prove that this sequence does not converge (again with respect to the metric \( d' \)).

Be careful with the following: it *seems* that if the sequence \( \{f_n\}_{n \in \mathbb{N}} \) converges, it should converge to the function

\[
f(x) = \begin{cases} 
0, & x \in \left[0, \frac{1}{2}\right) \\
1, & x \in \left[\frac{1}{2}, 1\right]
\end{cases}
\]

which is not continuous. But this is NOT a proof that the sequence \( \{f_n\}_{n \in \mathbb{N}} \) is not convergent with respect to the metric \( d' \): it could still be the case that there is some crazy (i.e., impossible to describe with a formula as above), continuous function to which the sequence converges. We have to discard this possibility.

Suppose the sequence \( \{f_n\}_{n \in \mathbb{N}} \) converges to a function \( g \in C[0, 1] \) with respect to the metric \( d' \). Then the following holds:

1. for all \( n \in \mathbb{N} \) and all \( x \in [0, 1] \),

\[
|f(x) - g(x)| = |f(x) - f_n(x) + f_n(x) - g(x)| \leq |f(x) - f_n(x)| + |g(x) - f_n(x)|;
\]

2. for all \( n \in \mathbb{N} \), \( \int_0^1 |f(x) - f_n(x)| \, dx = \frac{1}{2n} \); and

3. if \( \alpha_n = \int_0^1 |g(x) - f_n(x)| \, dx \) for each \( n \in \mathbb{N} \), then the sequence \( \{\alpha_n\}_{n \in \mathbb{N}} \) is convergent with limit 0.

Thus, the sequence \( \{\beta_n = \frac{1}{2n} + \alpha_n\}_{n \in \mathbb{N}} \) also converges to 0, and we have for all \( n \)

\[
0 \leq \int_0^1 |f(x) - g(x)| \, dx \leq \int_0^1 |f(x) - f_n(x)| \, dx + \int_0^1 |g(x) - f_n(x)| \, dx = \beta_n
\]
This means that $g(x) = f(x)$ for all $x \in [0, 1]$ such that both $f$ and $g$ are continuous at $x$. In other words, $g(x) = f(x)$ for all $x \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, and this is impossible because $g$ is continuous.

8.1. **Properties of complete spaces.** We have already seen some examples of complete and non complete metric space, and now it is time to study general properties of complete spaces and their subsets.

**Lemma 8.12.** Let $(X, d_X)$ be a metric space and let $A \subseteq X$ be a complete metric subspace. Then $A$ is closed in $X$.

**Proof.** Let $x \in \overline{A}$. Then, there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq A$ converging to $x$. Since every convergent sequence is a Cauchy sequence and $A$ is complete, it follows that the sequence $\{x_n\}_{n \in \mathbb{N}}$ is convergent to an element of $A$, and hence $x \in A$. □

**Lemma 8.13.** Let $(X, d_X)$ be a complete metric space, and let $A \subseteq X$ be a closed subset. Then $A$ is a complete metric subspace.

**Proof.** Let $\{x_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $A$. In particular, this is also a sequence in $X$, and since $X$ is complete it follows that the sequence $\{x_n\}$ is convergent, with limit $x$. Since $A$ is closed, then $x \in A$. □

A more important result is the following, relating the notions of compactness and completeness.

**Proposition 8.14.** Let $(X, d_X)$ be a compact metric space. Then $(X, d_X)$ is complete.

**Proof.** Let $\{x_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $A$, and let $S \subseteq A$ be the following subset 

$$S = \{x_1, x_2, \ldots, x_n, \ldots\} \subseteq A.$$ 

We may assume that $S$ is infinite (if $S$ is finite, then it follows immediately that the sequence $\{x_n\}$ is convergent). The proof is a bit long, so it is better to divide it in shorter steps.

1. There exists an “accumulation point” $a \in A$: for each $\varepsilon > 0$, 

$$B_{a, \varepsilon} \cap S \neq \emptyset \text{ or } \{a\}.$$ 

Suppose otherwise that no such element $a \in A$ exists. Then, for each $x \in A$ there exists some $\varepsilon(x) > 0$ such that 

$$B_{x, \varepsilon(x)} \cap S = \emptyset \text{ or } \{x\}.$$ 

The collection of open balls $\{B_{x, \varepsilon(x)}\}_{x \in A}$ is an open cover of $A$, and it must contain a finite subcover since $A$ is compact: there exist $x_1, \ldots, x_k$ such that 

$$S \subseteq A \subseteq B_{x_1, \varepsilon(x_1)} \cup B_{x_2, \varepsilon(x_2)} \cup \ldots \cup B_{x_k, \varepsilon(x_k)}.$$ 

But we are assuming that $S$ is infinite, and also that the intersection $B_{x, \varepsilon(x)} \cap S$ is either empty or $\{x\}$ for all $x \in A$, and hence, the above is impossible!
(2) Let \( a \in A \) be an “accumulation point” as in (1). Then, \( \{x_n\} \) contains a subsequence \( \{x_{n(k)}\}_{k \in \mathbb{N}} \) which converges to \( a \).

Note that we can assume that \( a \notin S \) (otherwise, we can remove all the terms \( x_n \) in the sequence \( \{x_n\} \) satisfying \( x_n = a \)). This part of the proof is done by induction, so let first \( n(1) \in \mathbb{N} \) be such that

\[
x_{n(1)} \in B_{a,1} \cap S.
\]

Such an integer must exist by part (1). Suppose now that we have chosen integers \( n(1) < n(2) < \ldots < n(k) \) such that

\[
x_{n(i)} \in B_{a,\frac{1}{k+1}} \cap S \quad i = 1, \ldots, k,
\]

and let \( \varepsilon = \min\{\frac{1}{k+1}, d_X(a,x_1), d_X(a,x_2), \ldots, d_X(a,x_{n(k)})\} \). Since \( x_n \neq a \) for all \( n \), it follows that \( \varepsilon > 0 \), and then by part (1) there exists some \( n(k+1) \in \mathbb{N} \) such that

\[
x_{n(k+1)} \in B_{a,\varepsilon} \cap S \subseteq B_{a,\frac{1}{k+1}} \cap S.
\]

Furthermore, the choice of \( \varepsilon \) implies that \( n(k + 1) > n(k) \), and this completes the induction step.

Now, it is clear that the sequence \( \{x_{n(k)}\}_{k \in \mathbb{N}} \) converges to \( a \) since, for all \( N \in \mathbb{N} \), if \( n(k) \geq N \) then \( d_X(a,x_{n(k)}) < \frac{1}{k} \leq \frac{1}{N} \).

(3) The sequence \( \{x_n\}_{n \in \mathbb{N}} \) converges to \( a \) because it is a Cauchy sequence.

Let \( \varepsilon > 0 \). Then there exist

(1) \( N \in \mathbb{N} \) such that, if \( n, m \geq N \), then \( d_X(x_n,x_m) < \varepsilon \);

(2) \( R \in \mathbb{N} \) such that, if \( r \geq R \), then \( d_X(a,x_{n(r)}) < \varepsilon \).

Choose then \( r \in \mathbb{N} \) such that \( n(r) \geq N, R \). Then

if \( n \geq N \), then

\[
d_X(a,x_n) \leq d_X(a,x_{n(r)}) + d_X(x_{n(r)},x_n) < 2 \cdot \varepsilon
\]

and hence the sequence \( \{x_n\}_{n \in \mathbb{N}} \) converges to \( a \).

Recall from Proposition 7.11 that, if \( A \subseteq X \) is a compact subset of the metric space \( (X,d_X) \), then \( A \) is also closed and bounded. Hence, we deduce the following.

**Corollary 8.15.** Let \( (X,d_X) \) be a metric space and let \( A \subseteq X \) be a compact subset. Then \( A \) is closed, bounded and complete.

The converse of the above corollary is not true, and a counter-example has already appeared in this chapter: the set \( \mathbb{N} \) with the discrete metric is closed (every metric space is closed), it is bounded (since \( \mathbb{N} \) is contained in the open ball of center 1 and radius 2), and it is complete (because it has the discrete metric); on the other hand, we have already seen in the tutorials that \( \mathbb{N} \) with the discrete metric is not compact. Nevertheless, this is almost what we are after.
**Definition 8.16.** Let \((X, d_X)\) be a metric space. A subset \(A \subseteq X\) is **totally bounded** if for each \(\varepsilon > 0\) there exists a finite set \(S_\varepsilon \subseteq A\) such that
\[
A \subseteq \bigcup_{x \in S_\varepsilon} B_{x, \varepsilon}.
\]

**Exercise 8.17.** Let \((X, d_X)\) be a metric space. Show that if \(A \subseteq X\) is totally bounded then it is bounded. Give an example of a metric space which is bounded but not totally bounded.

The following result is a refinement of something that we already knew: if a subset is compact, then it is bounded.

**Lemma 8.18.** Let \((X, d_X)\) be a metric space, and let \(A \subseteq X\) be a compact subset. Then, \(A\) is totally bounded.

*Proof.* Let \(\varepsilon > 0\), and consider the open cover \(\{B_{a, \varepsilon}\}_{a \in A}\). Since \(A\) is compact, this open cover must contain a finite subcover. In other words, there exist \(a_1, \ldots, a_k\) such that \(A \subseteq B_{a_1, \varepsilon} \cup \ldots \cup B_{a_k, \varepsilon}\). \(\Box\)

**Corollary 8.19.** Let \((X, d_X)\) be a metric space, and let \(A \subseteq X\) be a compact subset. Then, \(A\) is complete and totally bounded.

*Proof.* \(A\) is complete (Proposition 8.14), and \(A\) is totally bounded (Lemma 8.18). \(\Box\)

**Theorem 8.20.** Let \((X, d_X)\) be a metric space, and let \(A \subseteq X\) be a complete, totally bounded subset. Then \(A\) is compact.

*Proof.* This proof is also done by contradiction, so suppose that \(\{U_\gamma\}_{\gamma \in \Gamma}\) is an open cover of \(A\) which does not contain any finite subcover.

Since \(A\) is totally bounded, in particular there exists a finite set \(S_1 \subseteq A\) such that
\[
A \subseteq \bigcup_{x \in S_1} B_{x, 1}.
\]
Suppose that, for each \(x \in S_1\), the subset \(A \cap B_{x, 1}\) could be covered by a finite number of subsets \(U_\gamma\) in the open cover. Then, \(\{U_\gamma\}\) should contain a finite subcover, and we are assuming this is not the case. Hence, there exists \(y_0 \in S_1\) such that \(A \cap B_{y_0, 1}\) cannot be covered with a finite number of subsets \(U_\gamma\).

Since \(A\) is totally bounded and \(A \cap B_{y_0, 1} \subseteq A\), then \(A \cap B_{y_0, 1}\) is again totally bounded. In particular, there exists some finite subset \(S_2 \subseteq A \cap B_{y_0, 1}\) such that
\[
A \cap B_{y_0, 1} \subseteq \bigcup_{x \in S_2} B_{x, \frac{1}{2}}.
\]
Note also that \(d_X(y_0, x) \leq 1 + \frac{1}{2}\) for all \(x \in S_2\) by the triangle inequality.

Again, suppose that, for each \(x \in S_2\), the subset \(A \cap B_{x, \frac{1}{2}}\) could be covered by a finite number of subsets \(U_\gamma\) in the open cover. Then, \(A \cap B_{y_0, 1}\) would also be covered by
a finite number of elements $U_\gamma$, which contradicts our assumptions. Hence, there exists some $y_1 \in S_2$ such that $A \cap B_{y_1, \frac{1}{2}}$ cannot be covered by a finite number of elements $U_\gamma$.

The assumption that $\{U_\gamma\}$ does not contain any finite subcover implies that this process goes on forever. In particular, at the $n$-th step we can choose the radius of the open balls to be $\frac{1}{2^n}$, and this for each $n \geq 0$. This way, we construct a sequence $\{y_n\}_{n \geq 0}$ such that

$$d_X(x_n, x_{n+1}) \leq \frac{1}{2^n} + \frac{1}{2^{n+1}},$$

and in particular the sequence $\{y_n\}$ is Cauchy (as an exercise, check that this is indeed the case).

Now, since $A$ is complete by hypothesis, the sequence $\{y_n\}$ must converge to an element $a \in A$. Let then $U_\omega \in \{U_\gamma\}_\Gamma$ be such that $a \in U_\omega$.

Since $U_\omega$ is an open subset, there exists some $\delta > 0$ such that

$$B_{a, \delta} \subseteq U_\omega.$$

Furthermore, there exists some $n \in \mathbb{N}$ such that

$$d_X(y_n, a) < \frac{\delta}{2} \quad \frac{1}{2^n} < \frac{\delta}{2}.$$

Putting everything together, we have

$$A \cap B_{y_n, \frac{1}{2^n}} \subseteq B_{y_n, \frac{1}{2^n}} \subseteq B_{a, \delta} \subseteq U_\omega,$$

which is impossible since we chose each $y_n$ so that $A \cap B_{y_n, \frac{1}{2^n}}$ cannot be covered a finite number of elements $U_\gamma$. $\square$

8.2. The completion of a metric space. There are many situations where working in a complete metric space makes life much nicer. Fortunately for us, given a metric space $(X, d)$ we can always produce a new metric space $(\tilde{X}, \tilde{d})$ which is complete and unique with respect to $X$ in a very sensible way. This process is called completing the metric space $(X, d)$. Let’s formalize this.

**Definition 8.21.** Let $(X, d)$ be a metric space. A completion of $(X, d)$ is a metric space $(\tilde{X}, \tilde{d})$, together with a map $\iota: X \to \tilde{X}$, such that

1. $(\tilde{X}, \tilde{d})$ is complete;
2. $\iota$ is an isometry; and
3. $\text{Cl}(\iota(X)) = \tilde{X}$.

The task is double: we have to prove that such a completion always exists, and that it is unique up to homeomorphism. But first, an example.
Lemma 8.24. The function \( \sim \) is well-defined. Thus, we have to make sure that the limit
\[
\lim_{n \to \infty} d(x_n, y_n) = 0.
\]

Exercise 8.23. Check that this is an equivalence relation.

Let now \((X, d)\) be a metric space. We can then consider the set of all Cauchy sequences in \(X\), which we will call \(C(X)\). Let \(X, d\) be a metric space. We can then consider the set of all Cauchy sequences \(\{x_n\}_{n \in \mathbb{N}}\) in \(X\), and we can define a relation in \(C(X)\):
\[
\{x_n\}_{n \in \mathbb{N}} \sim \{y_n\}_{n \in \mathbb{N}} \iff \lim_{n \to \infty} d(x_n, y_n) = 0.
\]

Example 8.22. Consider the set \(\mathbb{Q}\) with the Euclidean metric. Then, \(\mathbb{R}\) with Euclidean metric is a completion of \(\mathbb{Q}\): \(\mathbb{R}\) is complete, the inclusion of \(\mathbb{Q}\) in \(\mathbb{R}\) is an isometry, and the closure of \(\mathbb{Q}\) in \(\mathbb{R}\) is the whole set of real numbers \(\mathbb{R}\).

Before we do anything else, there is something we have to check: it is not clear that \(\tilde{d}\) is a well-defined function. Thus, we have to make sure that the limit \(\lim_{n \to \infty} d(x_n, y_n)\) in the definition exists and that it does not depend on a particular choice of representatives.

Lemma 8.24. The function \(\tilde{d}\) in (7) is well-defined.

Proof. Let \(x, y \in \tilde{X}\), and let \(\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}} \subset C(X)\) be some representatives. Then we can consider the sequence of real numbers \(\{d(x_n, y_n)\}_{n \in \mathbb{N}}\): it is a Cauchy sequence! Indeed, for any \(n, m \in \mathbb{N}\), we have
\[
|d(x_n, y_n) - d(x_m, y_m)| = |d(x_n, y_n) - d(y_n, x_m) + d(y_n, x_m) - d(x_m, y_m)| \leq |d(x_n, y_n) - d(y_n, x_m)| + |d(y_n, x_m) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m),
\]
and thus \(\{d(x_n, y_n)\}_{n \in \mathbb{N}}\) is a Cauchy sequence because both \(\{x_n\}\) and \(\{y_n\}\) are Cauchy sequences. Since \(\mathbb{R}\) is complete, it follows that \(\{d(x_n, y_n)\}\) is a convergent sequence and its limit exists.

Let’s check now that the definition of \(\tilde{d}\) does not depend on a particular choice of representatives for \(x\) and \(y\). Let then \(\{x'_n\}, \{y'_n\} \subset C(X)\) be different representatives. Then the triangle inequality implies that, for all \(n\),
\[
d(x'_n, y'_n) \leq d(x'_n, x_n) + d(x_n, y_n) + d(y_n, y'_n).
\]

By hypothesis, \(\{x_n\} \sim \{x'_n\}\) and \(\{y_n\} \sim \{y'_n\}\), and hence we have
\[
\lim_{n \to \infty} d(x'_n, y'_n) = \lim_{n \to \infty} d(x'_n, x_n) + \lim_{n \to \infty} d(x_n, y_n) + \lim_{n \to \infty} d(y_n, y'_n) = \lim_{n \to \infty} d(x_n, y_n).
\]
**Proposition 8.25.** The function $\tilde{d}$ defined above is a metric on $\tilde{X}$.

*Proof.* Clearly, $\tilde{d}(x, y) \geq 0$ for all $x, y \in \tilde{X}$. As an exercise, check that $\tilde{d}(x, y) = 0$ if and only if $x = y$. Symmetry and the triangle inequality follow just because $d$ is a metric and already satisfies these properties. \( \Box \)

Before we can prove that $(\tilde{X}, \tilde{d})$ is a completion of $(X, d)$ we are still missing one piece of data: a reasonable map $\iota : X \to \tilde{X}$. How can we map $X$ into $\tilde{X}$? The simpler the better: by using the most basic of all Cauchy (and convergent) sequences: for each $x \in X$, define

$$ \iota : X \to \tilde{X} $$

by mapping each $x \in X$ to the class of the constant sequence \( \{x_n = x\}_{n \in \mathbb{N}} \).

**Theorem 8.26.** Let $(X, d)$ be a metric space. Then, $(\tilde{X}, \tilde{d})$, together with the map $\iota$ above, is a completion of $(X, d)$.

*Proof.* Let’s check the conditions as they appear in Definition 8.21.

(i) $(\tilde{X}, \tilde{d})$ is complete.

Let $\{x_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $\tilde{X}$. We have to check that it is convergent. For each $n \in \mathbb{N}$, fix $\{x_{n,k}\}_{k \in \mathbb{N}} \in C(X)$ be a representative of $x_n$, and let $\{y_k = x_{k,k}\}_{k \in \mathbb{N}}$. The following picture may help you in following the proof.

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$\ldots$</th>
<th>$x_n$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$k \in \mathbb{N}$</td>
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</tr>
<tr>
<td>$x_{1,k}$</td>
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<td>$x_{3,k}$</td>
<td>$\ldots$</td>
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<td>$\vdots$</td>
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<td>$x_{1,3}$</td>
<td>$x_{2,3}$</td>
<td>$x_{3,3}$</td>
<td>$\ldots$</td>
<td>$x_{n,3}$</td>
<td>$x_{3,3}$</td>
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</tr>
<tr>
<td>$x_{1,2}$</td>
<td>$x_{2,2}$</td>
<td>$x_{3,2}$</td>
<td>$\ldots$</td>
<td>$x_{n,2}$</td>
<td>$x_{2,2}$</td>
<td></td>
</tr>
<tr>
<td>$x_{1,1}$</td>
<td>$x_{2,1}$</td>
<td>$x_{3,1}$</td>
<td>$\ldots$</td>
<td>$x_{n,1}$</td>
<td>$x_{1,1}$</td>
<td></td>
</tr>
</tbody>
</table>

$n \in \mathbb{N}$
By the triangle inequality we have

\[ d(y_{k,k}, y_{r,r}) \leq d(x_{k,k}, x_{k,r}) + d(x_{k,r}, x_{r,r}), \]

and it follows that \( \{y_k\} \) is a Cauchy sequence because \( \{x_{k,r}\}_{r \in \mathbb{N}} \) is a Cauchy sequence in \( X \) and \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( \tilde{X} \). Let then \( y \in \tilde{X} \) be the class of \( \{y_k\} \).

Again using the triangle inequality, we have

\[ d(x_{k,k}, x_{n,k}) \leq d(x_{kk}, x_{n,n}) + d(x_{n,n}, x_{n,k}), \]

and this implies that

\[ \tilde{d}(y, x_n) = \lim_{k \to \infty} d(x_{k,k}, x_{n,k}) \xrightarrow{n \to \infty} 0 \]

In other words, \( y \) is the limit of the sequence \( \{x_n\} \), and thus \( \tilde{X} \) is complete.

(ii) The map \( \iota(x) \) is an isometry.

Recall that \( \iota: X \to \tilde{X} \) sends each \( x \in X \) to the class of the constant sequence \( \{x_n = x\}_{n \in \mathbb{N}} \). Let then \( x, y \in \tilde{X} \). We have

\[ \tilde{d}(\iota(x), \iota(y)) = \lim_{n \to \infty} d(x_n, y_n) = \lim_{n \to \infty} d(x, y) = d(x, y). \]

(iii) \( \text{Cl}(\iota(X)) = \tilde{X} \).

We have to check that, for each \( x \in \tilde{X} \), there is a sequence in \( \iota(X) \) that converges to \( x \). Let \( \{y_n\}_{n \in \mathbb{N}} \in C(\tilde{X}) \) be a sequence representing \( x \), and let \( x_n = \iota(y_n) \) for each \( n \). The sequence \( \{x_n\} \) is clearly a sequence in \( \iota(X) \).

This way, we have

\[ \tilde{d}(x, x_n) = \lim_{k \to \infty} d(y_k, y_n) \xrightarrow{n \to \infty} 0 \]

since \( \{y_n\} \) is a Cauchy sequence. Hence, the closure of \( \iota(X) \) is \( \tilde{X} \).

We have just proved that for each metric space there is always a completion, and now we have to make sure that this completion is unique in a sensible way.

**Theorem 8.27.** Let \((X, d)\) be a metric space. Then, the completion \((\tilde{X}, \tilde{d})\) is unique up to homeomorphism.

**Proof.** Let \((X', d')\), together with a map \( \iota': X \to X' \), be another completion of \((X, d)\). We have to construct a homeomorphism between \((\tilde{X}, \tilde{d})\) and \((X', d')\).

For each \( x \in \tilde{X} \) let \( \{\iota(x_n)\}_{n \in \mathbb{N}} \) be a sequence in \( \iota(X) \) which converges to \( x \) (such a sequence exists because \( \text{Cl}(\iota(X)) = \tilde{X} \)). We can consider then the sequence \( \{\iota'(x_n)\}_{n \in \mathbb{N}} \) in \( X' \).

Since \( \iota' \) is an isometry and the sequence \( \{x_n\} \) is a Cauchy sequence, it follows that \( \{\iota'(x_n)\} \) is a Cauchy sequence in \( X' \). Furthermore, \((X', d')\) is complete by hypothesis,
and hence the sequence \( \{ \iota'(x_n) \} \) is convergent, and we can define a map \( h : \widetilde{X} \rightarrow X' \) by \( h(x) = \lim_{n \to \infty} \iota'(x_n) \).

To check that this is well-defined, let \( \{ y_n \}_{n \in \mathbb{N}} \) is another sequence in \( X \) such that \( \{ \iota(y_n) \}_{n \in \mathbb{N}} \) converges to \( x \). Then, since \( \iota \) is an isometry, we have
\[
\lim_{n \to \infty} d(x_n, y_n) = \lim_{n \to \infty} \widetilde{d}(\iota(x_n), \iota(y_n)) = \widetilde{d}(\lim_{n \to \infty} \iota(x_n), \lim_{n \to \infty} \iota(y_n)) = \widetilde{d}(x, x) = 0,
\]
and then, since \( \iota' \) is also an isometry, we have \( \lim_{n \to \infty} \iota'(x_n) = \lim_{n \to \infty} \iota'(y_n) \).

In fact, the map \( h : \widetilde{X} \rightarrow X' \) is an isometry. Indeed, let \( x, y \in \widetilde{X} \), and let \( \{ x_n \}, \{ y_n \} \) be sequences in \( X \) such that \( \{ \iota(x_n) \}, \{ \iota(y_n) \} \) converge to \( x, y \) respectively. Then,
\[
\begin{align*}
d'(h(x), h(y)) &= d'(\lim_{n \to \infty} \iota'(x_n), \lim_{n \to \infty} \iota'(y_n)) = d'(\lim_{n \to \infty} \iota'(x_n), \lim_{n \to \infty} \iota'(y_n)) = \\
&= \lim_{n \to \infty} d(x_n, y_n) = \lim_{n \to \infty} \widetilde{d}(\iota(x_n), \iota(y_n)) = \widetilde{d}(x, y).
\end{align*}
\]
This proves that \( h \) is a continuous map. Notice that the definition of \( h \) only uses the properties of \( (\widetilde{X}, \widetilde{d}) \) as a completion of \( (X, d) \), but not the explicit definition of \( (\widetilde{X}, \widetilde{d}) \).

This means that we can define the inverse \( h^{-1} \) just as we defined \( h \) itself, and it will be an isometry again. Thus, \( h \) is bijective, and both \( h \) and \( h^{-1} \) are continuous: \( h \) is a homeomorphism. \( \square \)
9. Interlude II

One way of introducing what comes next, perhaps a bit too strong, is that we have been doing too much work in the previous chapters. Well of course this is not true, it was absolutely necessary to understand how a metric determines certain properties, such as continuity, connectivity, etc.

However, maybe you have noticed this already, the metric has been playing more and more a decorative role. If it did not cross your mind before, think of it now: continuity has been defined in terms of open/closed subsets, connectivity is also defined in terms of open/closed subsets, compactness is defined in terms of open/closed subsets...

Furthermore, since practically the beginning of the course we know lots of examples of different metrics that define the same open/closed subsets. What is the point then in fixing a metric and having to stick to it? Continuity of functions, convergence of sequences, connectivity... all these properties do not depend on the metric but on the equivalence class of a metric.

Thinking more geometrically, let me introduce the Klein bottle. This is a subset of \( \mathbb{R}^4 \), defined as the solutions of the equation

\[
(x^2 + y^2 + z^2 + 2y - 1) \left[(x^2 + y^2 + z^2 - 2y - 1)^2 - 8z^2\right] + 16xz(x^2 + y^2 + z^2 - 2y - 1) = 0.
\]

This way, the Klein bottle becomes a metric space, with the Euclidean metric. You may have heard of this amazing surface, where inside cannot be told apart from outside (as opposed to an sphere, where it is pretty clear what inside and outside mean), perhaps you have seen this picture before

However, from the point of view of metric spaces this is not very useful. For instance, how can we use the above description to define continuous maps to other metric spaces? Even checking that this space is connected is already a rather complicated task.

In addition, the above picture is really nice, but it is hard to manipulate. Let’s find a simpler way of representing the Klein bottle. You may remember that a cylinder
can be constructed from a stripe by gluing together two opposite sides following the same direction, while a Moebius band can be constructed from the same stripe by gluing together two opposite sides following opposite directions.

We can represent the Klein bottle in a similar way, as represented in the following picture.

This representation is easier to manipulate, but still it does not make it much easier to define a metric on the Klein bottle. The natural thing to do seems to be to induce a metric from the Euclidean metric on the square, but since we are gluing together (identifying) parts of the square that were unrelated, this leads to all sorts of trouble.

There must be some easier way... or perhaps the problem is that we are trying so desperately to keep a metric that we have not considered that maybe we do not need the metric any more.

Before we start formalizing all this, let me give you another reason why we should definitely consider something more general than metric spaces. we have recently seen how to construct a complete metric space out of any metric space, a process that we called completion. The idea for completion was to create the smallest possible metric space which has the missing property (in this case, completeness) and contains the original space.

Well, compactness is an important property, so why not doing the same for non-compact metric spaces? Simply because there is no reasonable way to do it in the language of metric space (here, reasonable means that can be done without filling pages and pages of calculations and proofs).

Time to leave the nest...
10. **Topological spaces**

Roughly speaking, a metric $d$ on a set $X$ is a rule to say whether two points are close or far from each other, by means of an exact scalar. From this point of view, in a topological space we still want to tell whether two points are close or far from each other, but in a vaguer way. Here is the proper definition.

**Definition 10.1.** Let $X$ be a set. A topology on $X$ is a collection $\mathcal{T}$ of subsets of $X$ satisfying the following conditions

1. (T1) the total set, $X$, and the empty set, $\emptyset$, are elements of $\mathcal{T}$;
2. (T2) if $\{U_{\gamma}\}_{\gamma \in \Gamma}$ is a (possibly infinite) family of elements of $\mathcal{T}$, then
   $$\bigcup_{\gamma \in \Gamma} U_{\gamma} \in \mathcal{T};$$
3. (T3) if $\{U_1, \ldots, U_n\} \subseteq \mathcal{T}$ is a finite family of elements of $\mathcal{T}$, then
   $$\bigcap_{i=1}^{n} U_i \in \mathcal{T}.$$ 

A topological space is a pair $(X, \mathcal{T})$ where $\mathcal{T}$ is a topology on $X$.

Does this definition remind you of anything?

**Example 10.2.** Let $(X, d)$ be a metric space, and let $\mathcal{T}$ be the collection of open subsets of $X$ with respect to the metric $d$. Then, $\mathcal{T}$ is a topology on $X$ by Theorem 3.10, and $(X, \mathcal{T})$ is a topological space. In other words, every metric on $X$ defines a topology on $X$.

Furthermore, by definition, two metrics $d$ and $d'$ on $X$ are topologically equivalent if they define the same open subsets. Hence, topologically equivalent metrics on $X$ define exactly the same topology on $X$!

**Definition 10.3.** Let $(X, \mathcal{T})$ be a topological space. Then, $U \subseteq X$ is an open subset if $U \in \mathcal{T}$ and is a closed subset if $U^c = X \setminus U \in \mathcal{T}$.

Perhaps the best way to see that topological spaces are a more general notion than metric spaces is the following example.

**Example 10.4.** Let $X$ be any set, and let $\mathcal{T}_1$ be the collection of all the subsets of $X$, and let $\mathcal{T}_2 = \{\emptyset, X\}$. Then, both $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on $X$. More precisely,

1. $\mathcal{T}_1$ is called the discrete topology on $X$, while
2. $\mathcal{T}_2$ is called the indiscrete topology on $X$. 

The topology $\mathcal{T}_1$ is induced by the discrete metric, whereas the $\mathcal{T}_2$ is as far from the discrete topology as it can be, hence the name, and it is not induced by a metric.

For instance, if $X$ is a finite set, then we know that any metric on $X$ is topologically equivalent to the discrete metric (Exercise 3 in List 3). Thus, the only topology on $X$ that is induced by a metric is the discrete topology.

**Example 10.5.** Let $X = \{a, b, c\}$ and let $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, X\}$. Then, $\mathcal{T}$ is a topology on $X$ which is not induced by any metric. Notice the following: there don’t exist open subsets $B, C \in \mathcal{T}$ such that $b \in B$, $c \notin B$ and $c \in C$, $b \notin C$. In other words, the elements $b$ and $c$ are “at distance zero” in this topology (of course the last statement does not make any sense).

**Example 10.6.** Let $\mathcal{T}$ be the collection of subsets of $\mathbb{R}$ which, in addition to $\emptyset$ and $\mathbb{R}$, contains all the subsets of the form $(x, \infty)$, for $x \in \mathbb{R}$. Clearly, this is a topology on $\mathbb{R}$, with less open subsets than the Euclidean topology, since for example the subset $(0, 1)$ is not open in the topology $\mathcal{T}$.

This is another example of a topology which is not induced by a metric on $\mathbb{R}$. Indeed, suppose that there exists a metric $d$ on $\mathbb{R}$ which induces the above topology. Since open balls are in particular open subsets, for each $x \in \mathbb{R}$ and each $\varepsilon > 0$, we have

$$B_{x, \varepsilon} = \mathbb{R} \text{ or } B_{x, \varepsilon} = (y, \infty)$$

for some $y \in \mathbb{R}$. Furthermore, if $B_{x, \varepsilon} = \mathbb{R}$ for all $x \in \mathbb{R}$ and all $\varepsilon > 0$, then $d$ cannot induce the topology $\mathcal{T}$ on $\mathbb{R}$ (why?).

Thus, there exists at least one $x \in \mathbb{R}$ and some $\varepsilon > 0$ such that

$$B_{x, \varepsilon} = (y, \infty)$$

for some $y \in \mathbb{R}$. This way, for each $n \in \mathbb{N}$, we have

$$B_{x, \frac{\varepsilon}{2^m}} = (y_{n+1}, \infty) \subseteq B_{x, \frac{\varepsilon}{2^n}} = (y_n, \infty)$$

for some $y_n, y_{n+1} \in \mathbb{R}$. This means that, for any $z > x$ and all $n \in \mathbb{N}$,

$$d(x, z) < \frac{\varepsilon}{2^n},$$

which is impossible since $z \neq x$.

Let’s see another example of all the unexpected situations that can happen when dealing with topological spaces.

**Example 10.7.** This example is usually called the line with two origins. Let $X = \mathbb{R} \cup \{z\}$ (here $z$ denotes simply an extra element), and let

$$\begin{align*}
(1) \quad & \mathcal{T}_1 = \{U \subseteq \mathbb{R} \mid U \text{ is open with the Euclidean metric} \}; \\
(2) \quad & \mathcal{T}_2 = \{\emptyset\} \cup \{W = V \cup \{z\} \mid V \in \mathcal{T}_1 \text{ is such that } \{0\} \notin V\}.
\end{align*}$$


Finally, let $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$. In other words, an element of $\mathcal{T}$ is a union $U \cup W$, with $U \in \mathcal{T}_1$ and $W \in \mathcal{T}_2$. This is a topology in $X$ (check the details), and it has the following property: if $A, B \in \mathcal{T}$ are such that

- $0 \in A, z \notin A$; and
- $z \in B, 0 \notin B$,

then $A \cap B \neq \emptyset$. In other words, it is impossible to separate 0 from $z$!

It turns out that almost all the notions that we have defined for metric spaces can be translated to topological spaces, with a major exception (can you guess which?).

10.1. **Set theory revisited.** Let’s start by reviewing the basic notions about set theory that we already know for metric spaces. There are no surprises in this section, everything behaves as it already does for metric spaces, so most of the proofs are omitted.

**Definition 10.8.** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

- The *interior* of $A$, $\text{Int}(A)$, is the greatest open subset of $X$ contained in $A$.
- The *closure* of $A$, $\text{Cl}(A)$, is the smallest closed subset of $X$ which contains $A$.
- The *boundary* of $A$, $\partial(A)$, is the intersection $\text{Cl}(A) \cap \text{Cl}(X \setminus A)$.

**Lemma 10.9.** Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Then,

1. the *interior* of $A$ is the union of all the open subsets which are contained in $A$:

$$\text{Int}(A) = \bigcup_{U \in \mathcal{T}, U \subseteq A} U;$$

2. the *closure* of $A$ is the intersection of all the closed subsets which contain $A$:

$$\text{Cl}(A) = \bigcap_{V \in \mathcal{T}, A \subseteq V} V.$$

**Proof.** We prove part (i) and leave part (ii) as an exercise (you will need the equivalent of Proposition 3.12 for topological spaces). Let $W = \text{cup}_{U \in \mathcal{T}, U \subseteq A} U$. Since $\mathcal{T}$ is a topology, $W \in \mathcal{T}$ and $W \subseteq A$. This means that $W \subseteq \text{Int}(A)$. On the other hand, $\text{Int}(A) \in \mathcal{T}$ and $\text{Int}(A) \subseteq A$. By definition, $\text{Int}(A) \subseteq W$. □

The following three lemmas describe the basic properties of interior and closure of subsets, and related both notions. Their proofs are left as an exercise (again, you just have to check the corresponding results for metric space).

**Lemma 10.10.** Let $(X, \mathcal{T})$ be a topological space. Then the following holds:

1. a subset $A \subseteq X$ is open if and only if $\text{Int}(A) = A$;
(2) for every $A \subseteq X$, $\text{Int}(\text{Int}(A)) = \text{Int}(A)$;

(3) if $\text{Int}(A) \subseteq B \subseteq A$ then $\text{Int}(A) = \text{Int}(B)$;

(4) if $\{U_\alpha\}_{\alpha \in \Gamma}$ is a collection (possibly infinite) of subsets of $X$, then

$$\bigcup_{\alpha \in \Gamma} \text{Int}(U_\alpha) \subseteq \text{Int} \left( \bigcup_{\alpha \in \Gamma} U_\alpha \right) \quad \text{and} \quad \text{Int} \left( \bigcap_{\alpha \in \Gamma} U_\alpha \right) \subseteq \bigcap_{\alpha \in \Gamma} \text{Int}(U_\alpha);$$

(5) if $\{U_1, \ldots, U_n\}$ is a finite collection of subsets of $X$, then

$$\text{Int} \left( \bigcap_{i=1}^n U_i \right) = \bigcap_{i=1}^n \text{Int}(U_i);$$

Lemma 10.11. Let $(X, \mathcal{T})$ be a topological space. Then the following holds:

(1) a subset $A \subseteq X$ is closed if and only if $A = \text{Cl}(A)$;

(2) for every $A \subseteq X$, $\text{Cl}(\text{Cl}(A)) = \text{Cl}(A)$;

(3) if $A \subseteq B \subseteq \text{Cl}(A)$ then $\text{Cl}(A) = \text{Cl}(B)$;

(4) if $\{U_\alpha\}_{\alpha \in \Gamma}$ is a collection (possibly infinite) of subsets of $X$, then

$$\bigcup_{\alpha \in \Gamma} \text{Cl}(U_\alpha) \subseteq \text{Cl} \left( \bigcup_{\alpha \in \Gamma} U_\alpha \right) \quad \text{and} \quad \text{Cl} \left( \bigcap_{\alpha \in \Gamma} U_\alpha \right) \subseteq \bigcap_{\alpha \in \Gamma} \text{Cl}(U_\alpha);$$

(5) if $\{U_1, \ldots, U_n\}$ is a finite collection of subsets of $X$, then

$$\text{Cl} \left( \bigcup_{i=1}^n U_i \right) = \bigcup_{i=1}^n \text{Cl}(U_i);$$

Lemma 10.12. Let $(X, \mathcal{T})$ be a topological space. Then, for any subsets $A, B \subseteq X$,

(1) the closure of the complement of $A$ is the complement of the interior of $A$:

$$\text{Cl}(A^c) = \text{Cl}(X \setminus A) = X \setminus \text{Int}(A) = \left( \text{Int}(A) \right)^c;$$

(2) the interior of the complement of $B$ is the complement of the closure of $B$:

$$\text{Int}(B^c) = \text{Int}(X \setminus B) = X \setminus \text{Cl}(B) = \left( \text{Cl}(B) \right)^c;$$

(3) $\partial(\text{Cl}(A)) \subseteq \text{Cl}(A)$.

(4) $\partial(\text{Int}(B)) \cap \text{Int}(B) = \emptyset$; and
11. Dealing with topological spaces

We want to study properties of topological spaces, the same way we wanted to understand how metric spaces behave. Many of the notions that we have studied for metric spaces make perfect sense when we consider topological spaces, and the only thing required is to translate properly the notion to the language of topological spaces. The idea of this section is to see which properties generalize to topological spaces, and how do they generalize.

This is, however, a difficult task without any further tools, so before we start analyzing general properties of topological spaces, let us develop a little bit this notion. Keep in mind Example 10.2, since it will inspire most of the definitions and results, and will give deeper meaning to anything said in this chapter.

First of all, if the objects to study are topological spaces, how do we compare two topological spaces? When we want to compare two groups or two rings, we use the notion of group or ring homomorphism. Thinking this way, we need to define what a “morphism” between topological spaces is.

**Definition 11.1.** Let \((X, \mathcal{T}_X)\) and \((Y, \mathcal{T}_Y)\) be topological spaces. A map \(f : X \to Y\) is a **continuous map** if, for all \(U \in \mathcal{T}_Y\),

\[
f^{-1}(U) \in \mathcal{T}_X.
\]

**Remark 11.2.** As already happened in Theorem 4.11, the notation \(f^{-1}(U)\) stands for the following subset of \(X\):

\[
f^{-1}(U) = \{ x \in X \mid f(x) \in U \}.
\]

No inverse function is assumed! Check Theorem 4.11 and the comments and remarks after it, because they apply to this situation as well. In particular, it is important to emphasize the following: if \(f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)\) is a continuous map, then

\[
W \in \mathcal{T}_X \implies f(W) \in \mathcal{T}_Y.
\]

**Proposition 11.3.** Let \(f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)\) and \(g : (Y, \mathcal{T}_Y) \to (Z, \mathcal{T}_Z)\) be continuous maps. Then, the composition \(g \circ f : (X, \mathcal{T}_X) \to (Z, \mathcal{T}_Z)\) is continuous.

**Proof.** For any \(W \in \mathcal{T}_Z\), we have \(U = (g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) \in \mathcal{T}_X\) because \(V = g^{-1}(W) \in \mathcal{T}_Y\) (\(g\) is continuous) and \(U = f^{-1}(V) \in \mathcal{T}_X\) (\(f\) is continuous). \(\square\)

If continuous maps are the equivalent of group homomorphisms for topological spaces, then the following correspond to isomorphisms between topological spaces.

**Definition 11.4.** Let \((X, \mathcal{T}_X)\) and \((Y, \mathcal{T}_Y)\) be topological spaces. A map \(f : X \to Y\) is a **homeomorphism** if \(f\) is bijective (as a map between sets) and both \(f\) and \(f^{-1}\) are continuous.
Remark 11.5. And what happens with isometries? Recall that an isometry is a map $f: (X, d_X) \to (Y, d_Y)$ such that $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$. Well, this notion simply does not make much sense when we deal with topological spaces.

The definition of topology is a natural generalization of the notion of metric, but it is not easy to check conditions (T1) - (T2) - (T3) in all cases. Perhaps it is a good idea to carefully analyze Example 10.2.

Let $(X, d)$ be a metric space, and let $\mathcal{O}_d$ be the collection of open subsets of $X$ with respect to the metric $d$. Within $\mathcal{O}_d$ there is a sub-collection of special importance: the open balls in $X$. It is true that the notion of open ball cannot be generalized to topological spaces, since the notion of radius means nothing now.

**WARNING!** As a consequence of this, the notion of bounded subset has little meaning when working with topological spaces.

In any case it is worth analyzing the behavior of the collection of open balls as a subset of $\mathcal{O}_d$. Recall the definition of open subset: $U \in \mathcal{O}_d$ if, for each $x \in U$ there exists $\varepsilon(x) > 0$ such that $U$ contains the open ball of radius $\varepsilon(x)$ around $x$: $B_{x, \varepsilon(x)} \subseteq U$. Another way of putting this is

$$U = \bigcup_{x \in U} B_{x, \varepsilon(x)},$$

so it makes sense to say that the collection of all open balls in $X$ generate all the open subsets in $\mathcal{O}_d$.

**Definition 11.6.** Let $(X, \mathcal{T})$ be a topological space. A **basis** for $\mathcal{T}$ is a collection $\mathcal{B} \subseteq \mathcal{T}$ such that each open set is a union of sets in $\mathcal{B}$.

**WARNING!** The notion of basis of a topology does NOT have the same kind of properties as, for instance, the basis of a vector space. As we will see in the following example, the same topology may have many, different bases!

**Example 11.7.** Consider $\mathbb{R}$ with the topology induced by the Euclidean metric, and let

$$\mathcal{B}_1 = \{(x_0, x_1) \subseteq \mathbb{R} \mid x_0, x_1 \in \mathbb{Q}\} \quad \mathcal{B}_2 = \{(x_0, x_1) \subseteq \mathbb{R} \mid x_0, x_1 \in \mathbb{R} \setminus \mathbb{Q}\}$$

Both collections above are bases for the same topology, their intersection is empty, and while $\mathcal{B}_1$ has countably many elements, $\mathcal{B}_2$ does not!

An alternative characterization of the basis of a topology is as follows.

**Lemma 11.8.** Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B}$ be a subset of $\mathcal{T}$. Then, $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if it satisfies the following properties.

(B1) For each $U \in \mathcal{T}$ and each $x \in U$ there exists some $V \in \mathcal{B}$ such that $x \in V \subseteq U$.

(B2) Let $V_1, V_2 \in \mathcal{B}$ and let $x \in V_1 \cap V_2$. Then there exists $V \in \mathcal{B}$ such that $x \in V \subseteq V_1 \cap V_2$. 
Proof. Suppose first that $B$ is a basis for $T$. In order to check condition (B1), let $U \in T$ and $x \in U$. By definition of basis, there exists some collection $\{V_\gamma\}_{\gamma \in \Gamma} \subseteq B$ such that

$$U = \bigcup_{\gamma \in \Gamma} V_\gamma.$$  

In particular, there is some $V_\gamma$ such that $x \in V_\gamma \subseteq U$.

Let now $V_1, V_2 \in B$ and let $x \in V_1 \cap V_2$. Since $V_1, V_2 \in T$ and $T$ is a topology, it follows that $V_1 \cap V_2 \in T$. Thus, by property (B1), there exists $V \in B$ such that $x \in V \subseteq V_1 \cap V_2$, and property (B2) holds.

Suppose now that $B$ satisfies properties (B1) and (B2) above. We have to check that each element $U \in T$ is a union of elements of $B$. Fix an open subset $U \in T$. For each $x \in U$ there exists some $V_x \in B$ satisfying condition (B1), and we can write

$$U = \bigcup_{x \in U} V_x.$$  

Hence $B$ is a basis for $T$. □

We can use Lemma 11.8 to formalize this idea of a topology generated by a basis. Let $X$ be a set and let $B$ be a collection of subsets of $X$ satisfying

(B1') For each $x \in X$ there exists some $V \in B$ such that $x \in V$.

(B2) Let $V_1, V_2 \in B$ and let $x \in V_1 \cap V_2$. Then there exists $V \in B$ such that $x \in V \subseteq V_1 \cap V_2$.

Now we can define the following collection of subsets:

$$T_B = \{ U \subseteq X \mid \text{for each } x \in U \text{ there exists } V \in B \text{ such that } x \in V \subseteq U \}.$$  

**Proposition 11.9.** The collection of subsets $T_B$ is a topology on $X$ with basis $B$.

We call $T_B$ the **topology generated by** $B$. Again, keep in mind that the same topology may be generated by more than one basis.

**Proof.** First, let’s prove that $T_B$ is a topology on $X$.

(T1) Both $\emptyset$ and $X$ satisfy the condition to be elements of $T_B$.

(T2) Let $\{U_\gamma\}_{\gamma \in \Gamma} \subseteq T_B$. We have to check that $U = \cup_1 U_\gamma \in T_B$. Each element $x \in U$ is contained in $U_\gamma$, for some $\gamma \in \Gamma$. Now $U_\gamma \in T_B$, and this means that there exists some $V \in B$ such that

$$x \in V \subseteq U_\gamma \subseteq U.$$  

Thus, $U \in T_B$.  

(T3) Let \( \{U_1, \ldots, U_n\} \subseteq \mathcal{B} \). We have to show that \( U = \cap_i U_i \in \mathcal{T}_B \). Let then \( x \in U \). This means that \( x \in U_i \) for all \( i \), and thus there exist \( V_1, \ldots, V_n \in \mathcal{B} \) such that \( x \in V_i \subseteq U_i \) for each \( i = 1, \ldots, n \).

Since \( \mathcal{B} \) satisfies condition (B2), it follows that there exists some \( V \in \mathcal{B} \) such that \( x \in V \subseteq U_i \) for each \( i = 1, \ldots, n \).

Thus, \( x \in V \subseteq U \) and \( U \in \mathcal{T}_B \).

The fact that \( \mathcal{B} \) is a basis for \( \mathcal{T}_B \) is a consequence of Lemma 11.8.

The following exercises were already suggested for metric spaces.

**Exercise 11.10.** Let \( f : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y) \) be a map, and let \( \mathcal{B} \subseteq \mathcal{T}_Y \) be a basis. Show that \( f \) is continuous if and only if \( f^{-1}(V) \in \mathcal{T}_X \) for each \( V \in \mathcal{B} \).

**Exercise 11.11.** Let \( f : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y) \) be a continuous map, and let \( U \subseteq X \). Show that if \( f \) is a homeomorphism, then the following holds:

1. if \( U \in \mathcal{T}_X \) then \( f(U) \in \mathcal{T}_Y \); and
2. if \( U^c \in \mathcal{T}_X \) then \( f(U)^c \in \mathcal{T}_Y \).

11.1. **From metric spaces to topological spaces.** We have now the tools we need, and it is time to start translating from metric to topological spaces. As an example of what we are going to do in this section, consider the following situation. If \( (X, d) \) is a metric space and \( A \subseteq X \) is any subset, then the metric \( d \) restricts to a metric on \( A \), and this makes \( (A, d|_A) \) into a metric space. The metric \( d|_A \) is usually called the *subspace metric*.

Is it possible to mimic this process of restricting the metric \( d \) to a subset for topological spaces? In other words, does it make sense to restrict a topology to a subset? And if so, how do we do it? Understanding the example of the metric space \( (X, d) \) and the subspace \( (A, d|_A) \) gives the right hints.

Indeed, let \( \mathcal{O}_X \) and \( \mathcal{O}_A \) be the collections of open subsets of \( X \) and \( A \) respectively, with respect to the metrics \( d \) and \( d|_A \) respectively. Then, recall that each open subset of \( A \) is the intersection of an open subset of \( X \) with \( A \):

\[
U \in \mathcal{O}_A \text{ if and only if } \exists W \in \mathcal{O}_X \text{ such that } U = W \cap A.
\]

Now, let \( (X, \mathcal{T}) \) be a topological space, and let \( A \subseteq X \) be a subset. We can then define

\[
\mathcal{T}_A = \{ U \cap A \mid U \in \mathcal{T} \},
\]

and it is not difficult to check that this is a topology.

(T1) \( \emptyset = \emptyset \cap A, \quad A = X \cap A \in \mathcal{T}_A \).
(T2) Let \( \{ U_\gamma \}_{\gamma \in \Gamma} \subseteq \mathcal{T}_A \). By definition of \( \mathcal{T}_A \), for each \( \gamma \in \Gamma \) there exists some \( W_\gamma \in \mathcal{T} \) such that \( U_\gamma = A \cap W_\gamma \). Since \( \mathcal{T} \) is a topology, we have \( W = \bigcup_{\gamma \in \Gamma} W_\gamma \in \mathcal{T} \), and hence
\[
U = \bigcup_{\gamma \in \Gamma} U_\gamma = \bigcup_{\gamma \in \Gamma} (W_\gamma \cap A) = \left( \bigcup_{\gamma \in \Gamma} W_\gamma \right) \cap A = W \cap A \in \mathcal{T}_A.
\]

(T3) Let \( \{ U_1, \ldots, U_n \} \subseteq \mathcal{T}_A \). Again, for each \( i \) there exists some \( W_i \in \mathcal{T} \) such that \( U_i = W_i \cap A \). Since \( \mathcal{T} \) is a topology, we have
\[
U = \bigcap_{i=1}^n U_i = \bigcap_{i=1}^n (W_i \cap A) = \left( \bigcap_{i=1}^n W_i \right) \cap A \in \mathcal{T}_A.
\]

This is known as the \textit{subspace topology} for obvious reasons.

Now we can try something slightly more complicated. Let \((X, \mathcal{T}_X)\) and \((Y, \mathcal{T}_Y)\) be topological spaces, and consider the set \( X \times Y \). As happened with metric spaces, we want to put a topology on this product, in a sensible way.

Now, if you check the corresponding situation for metric spaces (Theorem 3.9) you will see that the “right” metric to put on a product was the strongest possible metric which induces the original metrics on each of the factors of the product. Something similar could be done for topological spaces, but we do not have the time.

Instead, let \((X, \mathcal{T}_X)\) and \((Y, \mathcal{T}_Y)\) be the topological spaces above, let \( X \times Y \) be the product set, and let \( \mathcal{T}_{X \times Y} \) be the collection of subsets of \( X \times Y \) generated by
\[
\mathcal{B}_{X \times Y} = \{ U \times V \mid U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Y \}.
\]

Let also \( \pi_X : X \times Y \to X \) and \( \pi_Y : X \times Y \to Y \) be the obvious projection functions.

**Theorem 11.12.** The collection \( \mathcal{T}_{X \times Y} \) is the only topology on \( X \times Y \) satisfying the following condition.

(P) A map \( f : (Z, \mathcal{T}_Z) \to (X \times Y, \mathcal{T}_{X \times Y}) \) is continuous if and only if the compositions \( \pi_X \circ f : Z \to X \) and \( \pi_Y \circ f : Z \to Y \) are continuous.

**Proof.** The collection \( \mathcal{B}_{X \times Y} \) satisfies the conditions (B1) and (B2) in Lemma 11.8, and thus \( \mathcal{T}_{X \times Y} \) is the topology generated by \( \mathcal{B}_{X \times Y} \) (so it is a topology). We have to check then that it satisfies property (P), and that it is unique satisfying this condition. The rest of the proof is divided into several smaller parts.

1. The maps \( \pi_X : (X \times Y, \mathcal{T}_{X \times Y}) \to (X, \mathcal{T}_X) \) and \( \pi_Y : (X \times Y, \mathcal{T}_{X \times Y}) \to (Y, \mathcal{T}_Y) \) are continuous.

For instance, let’s check this for \( \pi_X \): we have to show that \( \pi_X^{-1}(U) \in \mathcal{T}_{X \times Y} \) for all \( U \in \mathcal{T}_X \), and this is immediate because \( \pi_X^{-1}(U) = U \times X \).

2. If \( f : (Z, \mathcal{T}_Z) \to (X \times Y, \mathcal{T}_{X \times Y}) \) is continuous then so are the compositions \( \pi_X \circ f \) and \( \pi_Y \circ f \).
This is an immediate consequence of Proposition 11.3: the composition of continuous maps is continuous.

3. If the compositions \( \pi_X \circ f : Z \to X \) and \( \pi_Y \circ f : Z \to Y \) are continuous then so is the map \( f : Z \to X \times Y \).

We have to check that, for each \( W \in \mathcal{T}_{X \times Y} \), \( f^{-1}(W) \in \mathcal{T}_Z \). Let’s assume first that \( W = U \times V \), for some \( U \in \mathcal{T}_X \) and \( V \in \mathcal{T}_Y \). Then we have

\[
f^{-1}(W) = f^{-1}(U \times V) = f^{-1}((U \times Y) \cap (X \times V)) = f^{-1}(U \times Y) \cap f^{-1}(X \times V) = f^{-1}(\pi_X^{-1}(U)) \cap f^{-1}(\pi_Y^{-1}(V)) = (\pi_X \circ f)^{-1}(U) \cap (\pi_Y \circ f)^{-1}(V) \in \mathcal{T}_Z.
\]

This means that the preimage of any element of the basis \( \mathcal{B}_{X \times Y} \) is in \( \mathcal{T}_Z \). Now, each \( W \in \mathcal{T}_{X \times Y} \) is a union of elements of \( \mathcal{B}_{X \times Y} \), and this shows that \( f \) is continuous.

4. The topology \( \mathcal{T}_{X \times Y} \) is unique satisfying property (P).

Let \( \mathcal{T} \) be another topology on \( X \times Y \) satisfying property (P), and consider the identity map \( \text{Id}: (X \times Y, \mathcal{T}) \to (X \times Y, \mathcal{T}) \). This is clearly continuous, and hence \( \pi_X = \pi_X \circ \text{Id} \) and \( \pi_Y = \pi_Y \circ \text{Id} \) are continuous too. This means that, for each \( U \in \mathcal{T}_X \) and \( V \in \mathcal{T}_Y \), we have

\[
\pi_X^{-1}(U) = U \times Y \in \mathcal{T} \quad \pi_Y^{-1}(V) = X \times V \in \mathcal{T}
\]

and hence \( U \times V = (U \times Y) \cap (X \times V) \in \mathcal{T} \). In other words, \( \mathcal{T} \) contains the basis \( \mathcal{B}_{X \times Y} \), and thus \( \mathcal{T}_{X \times Y} \subseteq \mathcal{T} \).

On the other hand, we know from part 1. that the maps

\[
\pi_X: (X \times Y, \mathcal{T}_{X \times Y}) \to (X, \mathcal{T}_X) \quad \pi_Y: (X \times Y, \mathcal{T}_{X \times Y}) \to (Y, \mathcal{T}_Y)
\]

are continuous. Notice that the compositions

\[
\pi_X: (X \times Y, \mathcal{T}_{X \times Y}) \xrightarrow{\text{Id}} (X \times Y, \mathcal{T}) \xrightarrow{\pi_X} (X, \mathcal{T}_X) \\
\pi_Y: (X \times Y, \mathcal{T}_{X \times Y}) \xrightarrow{\text{Id}} (X \times Y, \mathcal{T}) \xrightarrow{\pi_Y} (Y, \mathcal{T}_Y)
\]

are continuous maps, and thus by property (P) the map \( \text{Id}: (X \times Y, \mathcal{T}_{X \times Y}) \to (X \times Y, \mathcal{T}) \) is continuous, and this means that \( \mathcal{T} \subseteq \mathcal{T}_{X \times Y} \). \( \Box \)

**WARNING!** The definition of the product topology \( \mathcal{T}_{X \times Y} \) on \( X \times Y \) does NOT say that all the open subsets in \( \mathcal{T}_{X \times Y} \) are of the form \( U \times V \) for \( U \in \mathcal{T}_X \) and \( V \in \mathcal{T}_Y \)!

For instance, let \( \mathcal{T}_\mathbb{R} \) be the Euclidean topology (induced by the Euclidean metric) on \( \mathbb{R} \), and let \( \mathcal{T}_{\mathbb{R} \times \mathbb{R}} \) be the product topology on \( \mathbb{R} \times \mathbb{R} \). The subsets \( (0, 2), (1, 3) \subseteq \mathbb{R} \) are open in this topology, and we may consider the subset
\[ W = ([0, 2] \times [0, 2]) \cup ([1, 3] \times [1, 3]) \subseteq \mathbb{R} \times \mathbb{R} \]

which is not of the form \( U \times V \) for any \( U, V \in \mathcal{T} \)!

**Exercise 11.13.** Let \( f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y) \) be a continuous map, and let
\[
G_f = \{(x, f(x)) \in X \times Y \mid x \in X\} \subseteq X \times Y
\]
be the graph of the function, with the subspace topology induced from the product topology on \( X \times Y \). Show that \( G_f \) is homeomorphic to \( X \).

11.2. **A very basic metric-topological dictionary.** In the previous sections you have got a taste of how metric spaces “evolve” into topological spaces. Let’s see now how the main properties of metric spaces can be adapted to topological spaces. The first of these properties is connectivity. You can compare the following definition with Definition 6.1, see how many differences you find.

**Definition 11.14.** A topological space \((X, \mathcal{T}_X)\) is connected if it satisfies the following.

(C) If a subset \( A \subseteq X \) is both open and closed then either \( A = X \) or \( A = \emptyset \).

**Example 11.15.** Every set \( X \) with the indiscrete topology is connected.

**Example 11.16.** Let \( X = \{a, b, c\} \) and let \( \mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, X\} \). Then \((X, \mathcal{T})\) is connected: let \( A \subseteq X \) be a proper subset, \( A \neq \emptyset \). Then either \( A \) has one or two elements. Suppose in addition that \( A \) is open. Then, either \( A = \{a\} \) or \( A = \{a, b\} \). In any case, \( A \) cannot be closed because
\[
\{a\}^c = \{b, c\} \notin \mathcal{T} \quad \{a, b\}^c = \{c\} \notin \mathcal{T}.
\]

The following result is a generalization of Theorem 6.5. This is another example of a result in which the metric played a rather decorative role, and the proof will be left as a (really easy) exercise.

**Theorem 11.17.** Let \((X, \mathcal{T})\) be a topological space. Then, the following are equivalent.

1. \((X, \mathcal{T})\) is connected.
2. There does not exist subsets \( U, V \in \mathcal{T} \) such that \( U \cup V = X \) and \( U \cap V = \emptyset \).
3. There does not exist any continuous function \( f : (X, \mathcal{T}) \to (\{0, 1\}, \mathcal{T}_{\text{dis}}) \) which is surjective.
Proof. As promised, the proof is left as an exercise. □

**Corollary 11.18.** Let $f : (X, T_X) \rightarrow (Y, T_Y)$ be a continuous map of topological spaces, and suppose $(X, T_X)$ is connected. Then, $f(X) \subseteq Y$ is connected with the subspace topology.

**Corollary 11.19.** Let $f : (X, T_X) \rightarrow (Y, T_Y)$ be a homeomorphism of topological spaces. Then, $(X, T_X)$ is connected if and only if $(Y, T_Y)$ is connected.

The relation between connectivity and products of topological spaces behave exactly as happened for metric spaces. Although the proofs for topological spaces require little modification from the proof of Theorem 6.13, we will do them again to practice proving statements on topological spaces.

**Lemma 11.20.** Let $(X, T)$ be a topological space, and let $\{A_\gamma\}_{\gamma \in \Gamma}$ be a collection of connected subsets of $X$ such that

$$A_\gamma \cap A_\omega \neq \emptyset$$

for all $\gamma, \omega \in \Gamma$. Then, $A = \bigcup_{\gamma} A_\gamma$ is a connected subset of $X$.

Proof. Let $f : A \rightarrow \{0, 1\}$ be a continuous function, where $\{0, 1\}$ has the discrete topology. Since $A_\gamma$ is connected for all $\gamma \in \Gamma$, we have

$$f(A_\gamma) = \alpha_\gamma \in \{0, 1\}.$$ Let now $A_\omega$ be another set in the collection. Since $A_\gamma \cap A_\omega \neq \emptyset$ it follows that $f(A_\omega) = f(A_\gamma)$. Hence, the map $f$ cannot be surjective, since it has to be constant, and thus $A$ is connected. □

**Theorem 11.21.** Let $(X, T_X)$ and $(Y, T_Y)$ be topological spaces. Then, the product $(X \times Y, T_{X \times Y})$ is connected if and only if both $(X, T_X)$ and $(Y, T_Y)$ are connected.

Proof. If the product space is connected, then so are $(X, T_X)$ and $(Y, T_Y)$, since the projections $\pi_X : X \times Y \rightarrow X$ and $\pi_Y : X \times Y \rightarrow Y$ are both continuous maps.

Suppose otherwise that $(X, T_X)$ and $(Y, T_Y)$ are connected. Fix some element $a \in X$ and consider the sets

- $B = \{a\} \times Y \subseteq X \times Y$; and
- $C_y = X \times \{y\} \subseteq X \times Y$, where $y \in Y$.

Then $B$ is homeomorphic to $Y$, while $C_y$ is homeomorphic to $X$ for all $y \in Y$. In particular, each of the above subsets is connected.

For each $y \in Y$ we have $C_y \cap B \neq \emptyset$, and hence the subset $U_y = B \cup C_y$ is connected for all $Y$, by the previous Lemma. Consider then the collection of subsets $\{U_y\}_{y \in Y}$. For any $y, y' \in Y$, the intersection $U_y \cap U_{y'}$ is non-empty, and thus, again by the previous Lemma, $\bigcup_{y \in Y} U_y = X \times Y$ is a connected topological space. □
Remark 11.22. We could talk about path-connected topological spaces, just as we did for metric spaces (again, the definition for topological spaces requires barely any change with respect to the definition for metric spaces). The results (and proofs) that we saw for path-connected metric spaces, in particular Theorem 6.18, generalize in the most obvious way to topological spaces.

Definition 11.23. Let \((X, \mathcal{T})\) be a topological space. A connected component of \(X\) is a subset of \(X\) which is maximal with respect to the property of being connected. The set of connected components of \(X\) is denoted by \(\pi_0(X)\).

Let’s see now how compactness is defined for topological spaces. As we did for connectivity, the missing proofs are left as exercises (where probably the only difficulty that you will find is to locate in the previous chapters the corresponding result about metric spaces).

Be careful now because some basic results about compact subsets of metric spaces do not generalize to topological space. For instance, recall that bounded is a meaningless word in this context!

Definition 11.24. Let \((X, \mathcal{T})\) be a topological space and let \(A \subseteq X\) be any subset. An open cover of \(A\) in \(X\) is a collection \(\{U_\gamma\}_{\gamma \in \Gamma} \subseteq \mathcal{T}\) such that

\[ A \subseteq \bigcup_{\gamma \in \Gamma} U_\gamma. \]

A subcover of \(\{U_\gamma\}_{\gamma \in \Gamma}\) is a subcollection \(\{V_\omega\}_{\omega \in \Omega} \subseteq \{U_\gamma\}_{\gamma \in \Gamma}\) such that \(\subseteq \bigcup_{\Omega} U_\omega.\)

The subset \(A\) is compact if every open cover of \(A\) in \(X\) contains a finite subcover.

WARNING!. As happened already for metric spaces, showing that a subset of a topological space is compact means checking ALL possible open covers!

Example 11.25. Let \(X\) be any set. Then, \(X\) is compact with the indiscrete topology.

Example 11.26. Let \(X = \{a, b\}\) and \(\mathcal{T} = \{\emptyset, \{a\}, X\}\). Since \(X\) is finite, any of its subsets is compact. In particular, \(\{a\} \subseteq X\) is a compact subset, but it is not closed, since \(\{b\} \not\subseteq X\).

Proposition 11.27. Let \(f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)\) be a continuous map of topological spaces, and let \(A \subseteq X\) be a compact subset of \(X\). Then \(f(A) \subseteq Y\) is compact.

Corollary 11.28. Let \(f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)\) be a homeomorphism of topological spaces. Then, \(X\) is compact if and only if \(Y\) is compact.

Theorem 11.29. Let \((X, \mathcal{T}_X)\) and \((Y, \mathcal{T}_Y)\) be topological spaces, and let \(A \subseteq X\) and \(B \subseteq Y\). Then, \(A \times B \subseteq X \times Y\) is compact with respect to the product topology if and only if both \(A\) and \(B\) are compact.

It is a good exercise to do it again in the context of topological spaces: the proof follows the same lines as the proof of Theorem 7.17, but we have to replace the open balls by elements of the basis for the product topology.
Proof. If $A \times B$ is compact, then both $A$ and $B$ are compact since the projections $\pi_X: X \times Y \to X$ and $\pi_Y: X \times Y \to Y$ are continuous.

Suppose now that $A$ and $B$ are compact as subsets of $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ respectively, and let $\{W_\gamma\}_{\gamma \in \Gamma} \subseteq \mathcal{T}_{X \times Y}$ be an open cover of $A \times B$. Recall also that $\mathcal{T}_{X \times Y}$ is the topology generated by the basis
\[
\mathcal{B}_{X \times Y} = \{U \times V \subseteq X \times Y \mid U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Y\}.
\]

• We can assume that, for all $\gamma \in \Gamma$, $W_\gamma = U_\gamma \times V_\gamma \in \mathcal{B}_{X \times Y}$.

Let $W_\gamma$ be any term of the open cover. Since $\mathcal{B}_{X \times Y}$ is a basis of $\mathcal{T}_{X \times Y}$, for each $(x, y) \in W_\gamma$ there exists $K_x \times K_y \in \mathcal{B}_{X \times Y}$ such that
\[
(x, y) \in K_x \times K_y \subseteq W_\gamma,
\]
and we can write $W_\gamma = \bigcup_{(x,y)} K_x \times K_y$.

Doing this for each $\gamma \in \Gamma$, we end up with the open cover
\[
\{K_x \times K_y \mid (x, y) \in W_\gamma\}_{\gamma \in \Gamma}.
\]

If this new open cover contains a finite subcover then it follows that the original open cover $\{W_\gamma\}$ contains a finite subcover.

• If $W_\gamma = U_\gamma \times V_\gamma \in \mathcal{B}_{X \times Y}$ for all $\gamma \in \Gamma$, then the open cover $\{W_\gamma\}$ contains a finite subcover.

For each $a \in A$, consider the set $\{a\} \times B \subseteq A \times B$. This subset is homeomorphic to $B$, which is a compact subset of $Y$ by hypothesis, and thus $\{a\} \times B$ is compact too.

Since $\{W_\gamma\}$ is also an open cover of $\{a\} \times B$, it follows that there exists some finite subcover
\[
\Omega_a = \{W_{a,i_1}, \ldots, W_{a,i_n}\},
\]
where $W_{a,i_k} = U_{a,i_k} \times V_{a,i_k} \in \mathcal{B}_{X \times Y}$.

Still for $a \in A$, define the subset $U_a = U_{a,i_1} \cap U_{a,i_2} \cap \ldots \cap U_{a,i_n} \subseteq A$. This is an open subset of $X$ which contains the element $a$, and hence the collection $\{U_a\}_{a \in A}$ is an open cover of $A$. Since $A$ is compact there exist $a_1, \ldots, a_m \in A$ such that $\{U_{a_1}, \ldots, U_{a_m}\}$ is a finite subcover.

To finish the proof, notice that collection $\Omega_{a_1} \cup \ldots \cup \Omega_{a_m}$ is a finite subcover of $A \times B$. ☐

There is another property of metric spaces that we have studied in detail: completeness. Is there any sensible way to define completeness for topological spaces? The short answer is no: completeness requires that sequences behave very sensibly, and we have already seen in the tutorials that in general sequences in topological spaces behave pretty badly. A longer answer to the question would be, in fact, really long, so let’s take it as simply “no”.
12. What topological spaces can do that metric spaces cannot

In the previous chapter we did not see too many examples, it was a matter of redefining properties of metric spaces in the language of topological spaces, and there was few place for examples. The importance of topological spaces (and thus the interesting examples) lie in what we can do with them which we could not do with metric spaces.

In this chapter we will see two constructions which did not make much sense in the context of metric spaces, but which are natural in the context of topological spaces: quotient topological spaces and compactification of topological spaces. You can think of these constructions as geometric manipulations, and in this sense they are even more important since they are natural also as geometrical manipulations.

12.1. One-point compactification of topological spaces. The first construction to study has to do with compactness, and it has already appeared in the continuous assessment. Roughly speaking, the idea is the following: the property of compactness is missing in the topological space \((X, \mathcal{T}_X)\), so how can we modify \((X, \mathcal{T}_X)\) to make it compact? It turns out that it is enough to add an extra element to \(X\)!

Let’s formalize this. Let then \((X, \mathcal{T}_X)\) be a topological space, and define \(\tilde{X} = X \cup \{\infty\}\), where \(\infty\) simply designates an extra element which does not belong to \(X\). Now we have to define a topology on \(\tilde{X}\), and we want to do it so that \(\tilde{X}\) is compact and the restriction to \(X \subseteq \tilde{X}\) is the original topology \(\mathcal{T}_X\):

\[
\tilde{\mathcal{T}}_X = \mathcal{T}_X \cup \mathcal{T}',
\]

where \(\mathcal{T}' = \{\emptyset\} \cup \{W = V \cup \{\infty\} \subseteq \tilde{X} \mid V \subseteq X \text{ and } X \backslash V \text{ is compact and closed}\}\). This means that each element \(U \in \tilde{\mathcal{T}}_X\) can be written as

\[
U = Y \cup (V \cup \{\infty\}),
\]

where \(Y \in \mathcal{T}_X\) and \(V \cup \{\infty\} \in \mathcal{T}'\). The first thing to check is that \(\tilde{\mathcal{T}}_X\) is indeed a topology on \(\tilde{X}\).

**Lemma 12.1.** Let \((Y, \mathcal{T}_Y)\) be a topological space, and let \(B \subseteq Y\) be a compact subset. Then, each closed subset \(K \subseteq B\) is compact.

**Proof.** The proof is identical to the proof of Lemma 7.12. \(\Box\)

**Lemma 12.2.** The collection \(\tilde{\mathcal{T}}_X\) defined above is a topology on the set \(\tilde{X}\).

**Proof.** It is clear that \(\emptyset \in \tilde{\mathcal{T}}_X\), and \(\tilde{X} \in \tilde{\mathcal{T}}_X\) since \(X = \{\emptyset\} \cup (X \cup \{\infty\}) \in \mathcal{T}_X \cup \mathcal{T}' = \tilde{\mathcal{T}}_X\). That proves property (T1).
Let now \( \{ U_\gamma \}_{\gamma \in \Gamma} \subseteq \tilde{T}_X \). For each \( \gamma \in \Gamma \), we can write \( U_\gamma = Y_\gamma \cup W_\gamma \), with \( Y_\gamma \in \mathcal{T}_X \) and \( W_\gamma \in \mathcal{T}' \), and thus
\[
\bigcup_{\gamma \in \Gamma} U_\gamma = \left( \bigcup_{\gamma \in \Gamma} Y_\gamma \right) \cup \left( \bigcup_{\gamma \in \Gamma} W_\gamma \right).
\]

Since \( \mathcal{T}_X \) is a topology it follows that \( \cup_{\gamma} Y_\gamma \in \mathcal{T}_X \), and we have to check that \( \cup_{\gamma} W_\gamma \in \mathcal{T}' \). For simplicity, we can assume that \( W_\gamma \neq \emptyset \) for all \( \gamma \in \Gamma \). Thus, for each \( \gamma \),
\[
W_\gamma = V_\gamma \cup \{ \infty \},
\]
where \( V_\gamma \subseteq X \) is such that \( X \setminus V_\gamma \) is compact and closed, and we have to check that \( X \setminus (\cup_{\gamma} V_\gamma) \) is closed and compact.

Note that \( X \setminus V_\gamma \) is closed for each \( \gamma \in \Gamma \), and hence \( V_\gamma \in \mathcal{T}_X \). Thus, \( \cup_{\gamma} V_\gamma \in \mathcal{T}_X \), so \( X \setminus (\cup_{\gamma} V_\gamma) \) is closed. To show that \( X \setminus (\cup_{\gamma} V_\gamma) \) is compact, recall the equality
\[
X \setminus (\bigcup_{\gamma \in \Gamma} V_\gamma) = \bigcap_{\gamma \in \Gamma} (X \setminus V_\gamma).
\]
In particular, \( X \setminus (\cup_{\gamma} V_\gamma) \subseteq X \setminus V_\gamma \) for every \( \gamma \in \Gamma \), and by Lemma 12.1 it follows that \( X \setminus (\cup_{\gamma} V_\gamma) \) is compact. Thus, \( \cup_{\gamma} W_\gamma \in \mathcal{T}' \), and
\[
\bigcup_{\gamma \in \Gamma} U_\gamma \in \tilde{T}_X.
\]

Finally, let \( \{ U_1, \ldots, U_n \} \subseteq \tilde{T}_X \). We have to prove that the intersection \( \cap_{i=1}^n U_i \in \tilde{T}_X \).

Again, write \( U_i = Y_i \cup W_i \), with \( Y_i \in \mathcal{T}_X \) and \( W_i \in \mathcal{T}' \) for each \( i \). We have
\[
\bigcap_{i=1}^n U_i = \left( \bigcap_{i=1}^n Y_i \right) \cup \left( \bigcap_{i=1}^n W_i \right).
\]

Since \( \mathcal{T}_X \) is a topology, we have \( \cap_{i=1}^n Y_i \in \mathcal{T}_X \), and we have to check that \( \cap_{i=1}^n W_i \in \mathcal{T}' \). If \( W_i = \emptyset \) for any \( i \), then this is obvious, so suppose that \( W_i \neq \emptyset \) for all \( i \). We have
\[
\bigcap_{i=1}^n W_i = \bigcap_{i=1}^n (V_i \cup \{ \infty \}) = \left( \bigcap_{i=1}^n V_i \right) \cup \{ \infty \},
\]
where \( V_i \subseteq X \) is such that \( X \setminus V_i \) is closed and compact, for each \( i \). We have to check that \( X \setminus (\cap_{i=1}^n V_i) \) is closed and compact.

Note that \( X \setminus V_i \) is closed, and thus \( V_i \in \mathcal{T}_X \). Hence, \( \cap_{i=1}^n V_i \in \mathcal{T}_X \), and \( X \setminus (\cap_{i=1}^n V_i) \) is closed. To show that \( X \setminus (\cap_{i=1}^n V_i) \) is compact, let \( \{ A_\alpha \}_{\alpha \in \Lambda} \) be an open cover of \( X \setminus (\cap_{i=1}^n V_i) \), and recall the equality
\[
X \setminus \left( \bigcap_{i=1}^n V_i \right) = \bigcup_{i=1}^n (X \setminus V_i).
\]
In particular, \( \{ A_\alpha \}_{\alpha \in \Lambda} \) is an open cover of \( X \setminus V_i \), for each \( i \). Since \( X \setminus V_i \) is compact, there must exist a finite subcover, namely \( \Omega_i \). Putting all these finite subcovers
together we get \( \Omega_1 \cup \ldots \cup \Omega_n \subseteq \{A_\alpha\} \), which is a finite subcover of \( X \setminus (\cap_{i=1}^n V_i) \). Hence, \( \cap_{i=1}^n W_i \in \mathcal{T'} \), and thus
\[
\bigcap_{i=1}^n U_i \in \tilde{T}_X.
\]

Thus, \( \tilde{T}_X \) is a topology on \( \tilde{X} \). \(\square\)

**Theorem 12.3.** Let \( (X, \mathcal{T}_X) \) be a topological space, and let \( (\tilde{X}, \tilde{T}_X) \) be as above. Then, the following holds.

1. \( (\tilde{X}, \tilde{T}_X) \) is a compact space.
2. The restriction of \( \tilde{T}_X \) to \( X \) is \( \mathcal{T}_X \).

For obvious reasons, \( (\tilde{X}, \tilde{T}_X) \) is called the one-point compactification of \( (X, \mathcal{T}_X) \). It is sometimes called also Alexandroff’s compactification.

**Proof.** To prove (1), let \( \{U_\gamma\}_{\gamma \in \Gamma} \subseteq \tilde{T}_X \) be an open cover of \( \tilde{X} \), and write
\[
U_\gamma = Y_\gamma \cup W_\gamma
\]
with \( Y_\gamma \in \mathcal{T}_X \) and \( W_\gamma \in \mathcal{T'} \). For simplicity assume that \( W_\gamma \neq \emptyset \) for all \( \gamma \). Write then \( W_\gamma = V_\gamma \cup \{\infty\} \), where \( V_\gamma \subseteq X \) is such that \( X \setminus V_\gamma \) is closed and compact.

Now, since \( \{U_\gamma\} \) is an open cover of \( X \), there must exist some \( \omega \in \Gamma \) such that \( \infty \in U_\omega = Y_\omega \cup V_\omega \cup \{\infty\} \).

Notice that \( \{Y_\gamma \cup V_\gamma\}_{\gamma \in \Gamma} \) is now an open cover of \( X \setminus V_\gamma \), and the latter is compact by hypothesis. Thus, there is a finite subcover, namely \( \{Y_{\gamma_1} \cup V_{\gamma_1}, \ldots, Y_{\gamma_n} \cup V_{\gamma_n}\} \).

It follows then that \( \{U_{\gamma_1}, \ldots, U_{\gamma_n}, U_\omega\} \) is a finite subcover of \( \tilde{X} \), and hence it is compact.

To prove (2), let
\[
\mathcal{T}'_X = \{U \cap X \mid U \in \tilde{T}_X\},
\]
which is the topology on \( X \) induced by \( \tilde{T}_X \). We have to show that \( \mathcal{T}_X = \mathcal{T}'_X \). Clearly, \( \mathcal{T}_X \subseteq \mathcal{T}'_X \), because given \( Y \in \mathcal{T}_X \), we have \( Y \cup \{\emptyset\} \in \tilde{T}_X \).

Let then \( U \in \tilde{T}_X \). For simplicity we can assume that \( U = Y \cup (V \cup \{\infty\}) \), with \( Y \in \mathcal{T}_X \) and \( V \cup \{\infty\} \in \mathcal{T'} \). Then,
\[
X \cap U = (X \cap Y) \cup (X \cap V) \in \mathcal{T}_X,
\]
since \( Y, V, X \in \mathcal{T}_X \). Thus \( \mathcal{T}'_X \subseteq \mathcal{T}_X \). \(\square\)

Let’s see some examples of this construction.
**Example 12.4.** Let $X = \mathbb{R}$, with the topology induced from the Euclidean metric. Then, the one-point compactification of $X$ is the topological space in Exercise 3 of the second part of the continuous assessment. In particular, the one-point compactification of $\mathbb{R}$ is homeomorphic to a circle.

More generally, let $X = \mathbb{R}^n$, with the topology induced by the Euclidean metric. Then, the one-point compactification of $X$ is homeomorphic to the $n$-dimensional sphere $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_{n+1}^2 = 1\}$.

In view of the exercise in the continuous assessment, can you construct a homeomorphism from the $n$-dimensional sphere to the one-point compactification?

**Example 12.5.** The one-point compactification of the open interval $(0, 1)$ is, again, homeomorphic to the circle $S^1$. This is because $(0, 1)$ is homeomorphic to $\mathbb{R}$ (hence their one-point compactifications are homeomorphic).

12.2. **Quotient topological spaces.** This is the most geometrical of the two constructions, and it is actually rather important. Recall the idea of constructing a cylinder out of a stripe (a subset of $\mathbb{R}^2$), by gluing together two non-adjacent sides, as described in the following sketch.

In this case, it is easy to provide the cylinder with a metric (just consider the cylinder as a subset of $\mathbb{R}^3$ with the Euclidean metric). However, the example of the cylinder is part of a more general construction, so let’s see in more detail what is going on with the cylinder.

As a set, the cylinder is the quotient set of the stripe by the following equivalence relation:

$$(x, y) \sim (x, y) \text{ if } x \neq 0, 1 \quad (0, y) \sim (1, y).$$

The question is then which topology do we put on this quotient set. Notice that the quotient set of the stripe by this equivalence relation is not a subset of $\mathbb{R}^3$ (although it can be identified with a subset of $\mathbb{R}^3$). As happened with the product of topological spaces, not any topology is acceptable.

Let $(X, \mathcal{T})$ be a topological space, and let $\sim$ be an equivalence relation on the set $X$. You may think of the elements of $X$ in the same equivalence class as the elements
that we are going to glue together, just as we did with the stripe above. Let then \( \overline{X} \) be the set of equivalence classes, and let
\[
\pi: X \rightarrow \overline{X}
\]
be the map that sends each \( x \in X \) to its equivalence class. Notice that the map \( \pi \) is surjective (as a map of sets). Define also
\[
\mathcal{T} = \{ U \subseteq \overline{X} \mid \pi^{-1}(U) \in \mathcal{T} \}.
\]

**Lemma 12.6.** The collection \( \overline{\mathcal{T}} \) is a topology on \( \overline{X} \).

**Proof.** Since \( \emptyset = \pi^{-1}(\emptyset) \in \mathcal{T} \) and \( X = \pi^{-1}(\overline{X}) \), condition (T1) to be a topology is clearly seen to hold.

To check condition (T2), let \( \{ U_\gamma \}_{\gamma \in \Gamma} \subseteq \overline{\mathcal{T}} \). We have to check that the union \( \bigcup_{\gamma \in \Gamma} U_\gamma \in \overline{\mathcal{T}} \).

Notice that there is an equality
\[
f^{-1}\left( \bigcup_{\gamma \in \Gamma} U_\gamma \right) = \bigcup_{\gamma \in \Gamma} f^{-1}(U_\gamma).
\]

Since \( f^{-1}(U_\gamma) \in \mathcal{T} \) by hypothesis, and \( \mathcal{T} \) is a topology, we have \( \bigcup_{\gamma \in \Gamma} f^{-1}(U_\gamma) \in \mathcal{T} \), and thus \( \bigcup_{\gamma \in \Gamma} U_\gamma \in \overline{\mathcal{T}} \).

Finally, to check condition (T3), let \( \{ U_1, \ldots, U_n \} \subseteq \overline{\mathcal{T}} \). Again, there is an equality
\[
f^{-1}\left( \bigcap_{i=1}^{n} U_i \right) = \bigcap_{i=1}^{n} f^{-1}(U_i),
\]
and condition (T3) follows immediately. \( \square \)

For obvious reasons, the topological space \((\overline{X}, \overline{\mathcal{T}})\) is called the quotient space of \((X, \mathcal{T})\) by the equivalence relation \( \sim \), and the map \( \pi: X \rightarrow \overline{X} \) is called the quotient map.

**Theorem 12.7.** Let \((X, \mathcal{T})\) be a topological space, and let \( \sim \) be an equivalence relation on \( X \). Let also \((\overline{X}, \overline{\mathcal{T}})\) be the quotient space by \( \sim \). Then, \( \overline{\mathcal{T}} \) is the unique topology on \( \overline{X} \) satisfying the following conditions.

(Q1) The map \( \pi: (X, \mathcal{T}) \rightarrow (\overline{X}, \overline{\mathcal{T}}) \) is continuous.

(Q2) A map \( f: (\overline{X}, \overline{\mathcal{T}}) \rightarrow (Y, \mathcal{T}_Y) \) is continuous if and only if the composition \( f \circ \pi \) is continuous.

**Proof.** First we have to check that the topology \( \overline{\mathcal{T}} \) satisfies conditions (Q1) and (Q2). The map \( \pi: X \rightarrow \overline{X} \) is continuous since, by definition, \( \pi^{-1}(U) \in \mathcal{T} \) for all \( U \in \overline{\mathcal{T}} \).

Let then \( f: (\overline{X}, \overline{\mathcal{T}}) \rightarrow (Y, \mathcal{T}_Y) \). If \( f \) is continuous, then clearly \( f \circ \pi \) is continuous because it is a composition of continuous functions.
Suppose then that $f \circ \pi : (X, \mathcal{T}) \to (Y, \mathcal{T}_Y)$ is continuous. We have to show that $\pi^{-1}(V) \in \overline{\mathcal{T}}$ for all $V \in \mathcal{T}_Y$. Since $f \circ \pi$ is continuous, we have $(f \circ \pi)^{-1}(V) = f^{-1}(\pi^{-1}(V)) \in \mathcal{T}$, and hence by definition of $\mathcal{T}$ it follows that $\pi^{-1}(V) \in \overline{\mathcal{T}}$.

Now, suppose that $\overline{\mathcal{T}}'$ is another topology on $\overline{X}$ which also satisfies properties (Q1) and (Q2) with respect to the map $\pi' : (X, \mathcal{T}) \to (\overline{X}, \overline{\mathcal{T}})$. We have to prove that $\overline{\mathcal{T}}' = \overline{\mathcal{T}}$.

Consider first the identity map $\text{Id} : (X, \mathcal{T}) \to (X, \mathcal{T}')$. Since $\mathcal{T}'$ satisfies condition (Q1), the map $\pi' = \text{Id} \circ \pi$ is continuous and thus $\text{Id}$ is continuous by property (Q2). This implies that $\mathcal{T}' \subseteq \overline{\mathcal{T}}$.

Similarly, we can consider the map $\text{Id}' : (\overline{X}, \overline{\mathcal{T}}) \to (\overline{X}, \overline{\mathcal{T}}')$. Again, since $\pi = \text{Id} \circ \pi'$ is continuous, it follows that $\text{Id}'$ is continuous, and hence $\overline{\mathcal{T}} \subseteq \overline{\mathcal{T}}'$.

**Example 12.8.** How many ways do you know of “constructing” a topological space which is homeomorphic to the 2-dimensional sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}?$$

One way is the one-point compactification of the Euclidean plane. This way we obtain a huge 2-dimensional sphere, but a sphere all the same.

Another way is by gluing together two discs through their boundary, as sketched in the following picture.

![Diagram](URL)

Actually, it is enough to use a single circle.

![Single Circle Diagram](URL)
The above constructions, using circles, are examples of quotient spaces. In the first case, the set $X$ is a disjoint union of two disc, and the equivalence relation is given by identifying the boundary of one circle with the boundary of the other circle. In the second case, you can think of $X$ as the complex disc of radius 1, and we identify each element of the boundary with its complex conjugate.

Yet another way of constructing a 2-dimensional sphere, again using quotient spaces, is to “crush to a point” each of the lids of a cylinder.

And what about the Platonic solids? They are also homeomorphic to a sphere. In fact, any convex polyhedron is homeomorphic to a 2-dimensional sphere, and note that each polyhedron is in fact a quotient space, where $X$ is the disjoint union of all the faces, and we glue the faces together to form the polyhedron.
REFERENCES

The main published references for this course are the following.


Other general, electronic sources are the following.


References about the $p$-adic metric.


References about the Cantor set.

1. http://planetmath.org/cantorset

References about the Heine-Borel Theorem.


References about Hausdorff spaces (this has not been explicitly defined in the course, but the notion has appeared several times in examples).


References about the classification of surfaces.