MA1006 ALGEBRA – List 3 - SOLUTIONS

1. Read the wikipedia articles on a polynomial, a linear equation and other types of equations, on the Galois theory and about Gauss.

2. What was the doctoral dissertation of Gauss about?

3. Solve the following quadratic equations:
   a. \( x^2 + 2x + 1 = 0 \).
      The solution is \( x = -1 \).
   b. \( x^2 + x + 1 = 0 \).
      There are no solutions.
   c. \( 2x^2 + 3x + 4 = 0 \).
      There are no solutions.
   d. \( x^2 + ax + 2 = 0 \), where \( a \) is a real number.
      If \( a < 2\sqrt{2} \), there are no solutions. If \( a = 2\sqrt{2} \) then \( x = -\sqrt{2} \). If \( a > 2\sqrt{2} \), then
      \[ x = \frac{-a \pm \sqrt{a^2 - 8}}{2}. \]

4. For which values of \( t \in \mathbb{R} \) the following equations have (i) two distinct real solutions, (ii) a unique real solution, (iii) no real solutions:
   a. \( tx^2 + 3x - 5 = 0 \).
      If \( t > -\frac{9}{20} \), \( t \neq 0 \), there are two solutions. If \( t = 0 \) or \( t = -\frac{9}{20} \) then there is only one solution. Finally, if \( t < -\frac{9}{20} \) there is no solution.
   b. \( x^2 + tx - 5 = 0 \).
      There are two solutions for any value of \( t \).
   c. \( x^2 + 3x - t = 0 \).
      If \( t > -\frac{9}{4} \) there are two solutions. If \( t = -\frac{9}{4} \) then there is only one solution. If \( t < -\frac{9}{4} \) there is no solution.

5. Find two real numbers \( a \) and \( b \) such that their sum and their product are both equal to five.
   We have to solve the system of equations
   \[ \begin{cases} 
   a + b = 5 \\
   ab = 5 
   \end{cases} \]
   Notice that \( ab = 5 \) implies that both \( a, b \neq 0 \). Thus, \( b = \frac{5}{a} \). Replacing this in the first equation, we get \( a + \frac{5}{a} = 5 \), or equivalently \( a^2 + 5 = 5a \).
6. What is the smallest positive real number \( r \in \mathbb{R} \) such that there exists two real numbers \( a \) and \( b \) whose sum and product are both equal to \( r \)? What are \( a \) and \( b \) for this \( r \)?

Now we have to solve the system

\[
\begin{align*}
  a + b &= r \\
  ab &= r
\end{align*}
\]

Suppose for a second that \( r \neq 0 \). Then we can write \( b = \frac{r}{a} \), and replacing this in the first equation we get \( a + \frac{r}{a} = r \). The solutions of this equations are

\[
a = \frac{r \pm \sqrt{r^2 - 4r}}{2}.
\]

That is, the equation \( a^2 - ra + r = 0 \) has one or two solutions whenever \( r^2 - 4r \geq 0 \), and this implies that either \( r \leq 0 \) or \( r \geq 4 \).

Since we want the \textit{smallest positive} real number \( r \), the answer is \( r = 4 \). In this case, \( a = 2 = b \).

7. Divide (with remainder) each polynomial of degree at least two on this page by the polynomial:

a. \( x - 1 \):

- \( x^2 + 2x + 1 = (x - 1)(x + 3) + 4 \)
- \( x^2 + x + 1 = (x - 1)(x + 2) + 3 \)
- \( 2x^2 + 3x + 4 = (x - 1)(2x + 5) + 9 \)
- \( x^2 + ax + 2 = (x - 1)(x + (a + 1)) + a + 3 \)
- \( tx^2 + 3x - 5 = (x - 1)(tx + (t + 3)) + t - 2, \ (t \neq 0) \)
- \( x^2 + tx - 5 = (x - 1)(x + (t + 1)) + t - 4 \)
- \( x^2 + 3x - t = (x - 1)(x + 4) + 4 - t \)
- \( x^2 + 1 = (x - 1)(x + 1) + 2 \)
- \( x^3 - 3x^2 + 3x - 1 = (x - 1)(x^2 - 2x + 1) \)
- \( x^3 + 12x^2 + 47x + 60 = (x - 1)(x^2 + 13x + 60) + 120 \)
- \( x^4 + x^3 - 19x^2 + 11x + 30 = (x - 1)(x^3 + 2x^2 + 17x + 28) + 58 \)
- \( x^5 - x^4 - 4x^3 + 3x^2 + 3x - 2 = (x - 1)(x^4 - 4x^2 - x + 2) \)
- \( x^{10} + 2x^9 - 2x^8 + 4x^7 + x^6 + 5x^5 + 2x^4 + 8x^3 + x^2 - x + 8 = \)
  \[\]
  \[= (x - 1)(x^9 + 3x^8 + x^7 + 5x^6 + 6x^5 + 11x^4 + 13x^3 + 21x^2 + 22x + 21) + 29.\]
b. \(x^2 + 1;\)
   \(\bullet \quad x^2 + 2x + 1 = (x^2 + 1) + 2x\)
   \(\bullet \quad x^2 + x + 1 = (x^2 + 1) + x\)
   \(\bullet \quad 2x^2 + 3x + 4 = (x^2 + 1)2 + 3x + 2\)
   \(\bullet \quad x^2 + ax + 2 = (x^2 + 1) + ax + 1\)
   \(\bullet \quad tx^2 + 3x - 5 = (x^2 + 1)t + 3x - 5 - t, \ (t \neq 0)\)
   \(\bullet \quad x^2 + tx - 5 = (x^2 + 1) + tx - 6\)
   \(\bullet \quad x^2 + 3x - t = (x^2 + 1) + 3x - 1 - t\)
   \(\bullet \quad x^2 + 1 = x^2 + 1\)
   \(\bullet \quad x^3 - 3x^2 + 3x - 1 = (x^2 + 1)(x - 3) + 2x + 2\)
   \(\bullet \quad x^3 + 12x^2 + 47x + 60 = (x^2 + 1)(x + 12) + 46x + 48\)
   \(\bullet \quad x^4 + x^3 - 19x^2 + 11x + 30 = (x^2 + 1)(x^2 + x - 20) + 10x + 50\)
   \(\bullet \quad x^5 - x^4 - 4x^3 + 3x^2 + 3x - 2 = (x^2 + 1)(x^3 - x^2 - 5x + 4) + 8x - 6\)
   \(\bullet \quad x^{10} + 2x^9 - 2x^8 + 4x^7 + x^6 + 5x^5 + 2x^4 + 8x^3 + x^2 - x + 8 =
     (x^2 + 1)(x^8 + 2x^7 - 3x^6 + 2x^5 + 4x^4 + 3x^3 - 2x^2 + 5x + 3) - 6x + 5\)

8. Solve the following polynomial equations.

a. \(x^3 - 3x^2 + 3x - 1 = 0;\)
   There is only one solution, \(x = 1.\)

b. \(x^3 + 12x^2 + 47x + 60 = 0;\)
   The solutions are \(x = -5, \ x = -4\) and \(x = -3.\)

c. \(x^4 + x^3 - 19x^2 + 11x + 30 = 0;\)
   The solutions are \(x = -5, \ x = -1, \ x = 2\) and \(x = 3.\)

d. \(x^5 - x^4 - 4x^3 + 3x^2 + 3x - 2 = 0.\)
   The solutions are \(x = -1, \ x = 1, \ x = 2\) and \(x = \frac{-1 \pm \sqrt{3}}{2}.\)

9. Prove that the polynomial \(x^{10} + 2x^9 - 2x^8 + 4x^7 + x^6 + 5x^5 + 2x^4 + 8x^3 + x^2 - x + 8\) does not have rational roots. Can you remember, without checking the lecture notes, the general theorem about rational roots of polynomials?

If \(r = \frac{p}{q}\) is a rational root of this polynomial, with \(\text{gcd}(p, q) = 1,\) then we know that \(p\) divides 8 and \(q\) divides 1. Thus, the only possible rational roots of this polynomial are \(\pm 1, \pm 2, \pm 4, \pm 8.\) Since none of these numbers is a root (check it!), we deduce that this polynomial does not have any rational solution.
10. Let \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) be two polynomial functions defined by \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \) and \( g(x) = b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0 \), where \( a_i, b_i \in \mathbb{R} \). Observe that the composition \( g \circ f \) is also a polynomial function.

a. Compute several concrete examples in low degrees to gain intuition.

For instance, if \( f(x) = x^2 + 1 \) and \( g(x) = x - 1 \), then \((g \circ f)(x) = x^2\), while \((f \circ g)(x) = x^2 - 2x + 2\) (hence, composition of polynomials is not commutative!).

Another example: \( f(x) = x^2 + 3 \) and \( g(x) = 2x^2 - 3x + 4 \). Then,

\[
(g \circ f)(x) = g(f(x)) = g(x^2 + 3) = 2(x^2 + 3)^2 - 3(x^2 + 3) + 4 = 2(x^4 + 6x^2 + 9) - 3(x^2 + 3) + 4 = 2x^4 + 9x^2 + 13
\]

b. What is the degree of \( g \circ f \)?

\[
\deg(g \circ f) = \deg(g) \cdot \deg(f) = n^2.
\]

c. Compute the highest degree coefficient of \( g \circ f \) in terms of the coefficients of \( f \) and \( g \).

In \((g \circ f)(x)\) there is only one term \(x^{n^2}\), and its coefficient is \(a_n^2 b_n\).

d. Compute the free coefficient of \( g \circ f \) in terms of the coefficients of \( f \) and \( g \).

The formula for the free coefficient is \(\sum_{i=0}^{n} a_i b_i\).

e. Prove the following statement: if \( g \) has no real roots then \( g \circ f \) has no real roots.

Let \( P(x) = (g \circ f)(x) \) for short, and suppose that \( r \in \mathbb{R} \) is a root of \( P(x) \). That is, \( P(r) = 0 \). Then,

\[
0 = P(r) = (g \circ f)(r) = g(f(r)),
\]

and thus \( f(r) \) is a root of \( g \).

f. Prove that the following statement is false: if \( f \) has no real roots then \( g \circ f \) has no real roots.

The first example we tried (in this exercise) is already a counterexample for this statement: \( f(x) = x^2 + 1 \) has no real roots, but the composition of \( f(x) \) with \( g(x) = x - 1 \) is \((g \circ f)(x) = (x^2 + 1) - 1 = x^2\), and this polynomial has a real root, \( r = 0 \).

Summary of Week 3:

- Polynomial equations.
- Long division of polynomials.
- A theorem on rational solutions of polynomial equations.