The Restricted 3-Body Problem: A Mission to L4

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Abstract

The investigation sought to find a cost efficient route for a spacecraft to travel using Lagrangian Equilibrium points in the sun-earth system. By expressing the restricted three body problem as a system of ordinary differential equations and linearization we understand the behavior of points geometrically. Using these methods and a program which approximates solutions in systems of differential equations, we were able to find an orbit which uses the unstable nature of L1 to shoot off towards L4.

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This summer, July 21, 2011, marked the end of NASA’s space shuttle missions. Part of NASA’s plan for the future includes sending more unmanned spacecraft to study the Moon, Mars, and the outer planets. As part of this next stage, NASA is investigating ways to save fuel [?]. One way NASA could do this is by using the gravitational field of the earth and the sun, and ultimately other planets, to pull the spacecraft where they want it to go. The goal of our project was to find a fuel-saving path, using the chaotic nature of the system, by which a spacecraft could travel from an Earth orbit to L4, a point in space that forms an equilateral triangle with Earth and the Sun.

This problem is interesting for several reasons. Firstly, L4 is an interesting point to travel to because it is an equilibrium point of the ordinary differential equations that govern the motion of a spacecraft near the sun and the earth. Also, some small objects have collected at this point and could be interesting to study. Secondly, using chaos is interesting because it could cut down on the amount of fuel needed since we are letting gravity do most of the work instead of relying
heavily on the spacecraft’s propulsion system, which burns fuel. This is very important because getting fuel into space for spacecrafts is very expensive, about $10,000 per pound of fuel [?]. Lastly, this problem is interesting because in the last 20 years NASA has tried to incorporate some of these same ideas in planning space missions. The Japanese satellite Hiten, after completing its original mission, was sent on a follow-on mission to pass through the L4 of the Earth and Moon system. Hiten was also the first to exploit the low energy chaotic orbits like the one we considered [?]. These types of paths could easily become more and more important in the near future as NASA plans to send spacecraft to the Moon, Mars, and the outer planets in the next several years.

The result of our project is a trajectory for a spacecraft that starts in Earth orbit, uses its propulsion system to give it a small push, slingshots off an unstable equilibrium point in the system, and goes toward L4, where it must use a small amount of energy to slow it down so that it can orbit at L4.

We used MATLAB ode45 to find approximate solutions to our system of ordinary differential equations. We also looked at Jacobian Integrals to analyze the energy and the linearization of the system to analyze stability and instability.

2 Ordinary Differential Equations

Before we discuss the 3-body Problem and the Restricted 3-body Problem it is necessary to recall some facts about ordinary differential equations (ODEs).

2.1 Existence and Uniqueness

First, let us recall one of the most fundamental theorems of differential equations, The existence and uniqueness theorem.

**Theorem 1.** Given any initial condition \((t_0, x_0)\) for the equation \(\dot{x} = g(t, x)\), there exists a unique solution with \(x(t_0) = x_0\).

Given initial conditions \((t_0, x_0, \dot{x}_0)\) for the equation \(\ddot{x} = g(t, x, \dot{x})\), there exists a unique solution with \(x(t_0) = x_0\) and \(\dot{x}(t_0) = \dot{x}_0\).

Similarly, for systems we have the following theorem.

**Theorem 2.** \(\dot{\vec{x}} = \vec{f}(t, \vec{y})\) has a unique solution with \(\vec{x}(t_0) = \vec{x}_0\).

\(\ddot{\vec{x}} = \vec{f}(t, \dot{\vec{x}}, \vec{x})\) has a unique solution with \(\ddot{\vec{x}}(t_0) = \ddot{\vec{x}}_0\) and \(\dot{\vec{x}}(t_0) = \dot{\vec{x}}_0\)
2.2 Phase Plane Portraits

Now we will discuss a few terms and ideas that will be helpful when we later discuss the Restricted 3-body Problem. Note that $\vec{f}: \mathbb{R}^n \to \mathbb{R}^n$ can be thought of as a vector field on $\mathbb{R}^n$ and the solutions of the ODE are parallel to the vector field.

**Definition 3.** A phase plane portrait is a sketch of the images of some solutions. The time dependence cannot be seen from the phase plane.

2.2.1 Example

Consider a simple pendulum described by the second order differential equation $\ddot{\theta} + \sin(\theta) = 0$. This equation can be written as the following system of two first order differential equations.

$$
\begin{align*}
\dot{\theta} &= v \\
\dot{v} &= -\sin(\theta)
\end{align*}
$$

For this system, the vector field $\vec{f}: \mathbb{R}^2 \to \mathbb{R}^2$ is

$$
\vec{f}(\begin{bmatrix} \theta \\ v \end{bmatrix}) = \begin{bmatrix} v \\ -\sin(\theta) \end{bmatrix}.
$$

Figure 1 is an example of a phase plane portrait for the simple pendulum problem. The solutions shown are the solutions for initial conditions $\theta \in [-\pi, \pi]$ with $v = 0$. 

Figure 1: Note that the phase plane portrait graphs the position, $\theta$, with respect to the velocity, $v$. 
2.3 Equilibrium Points

Definition 4. We say that $\vec{x}_0 \in \mathbb{R}^n$ is an equilibrium point if

$$\vec{f}(\vec{x}_0) = 0$$

Lemma 5. If $\vec{x}_0$ is an equilibrium point, then $\vec{x}(t) = \vec{x}_0$ is a solution

Proof. $\vec{x}_0$ is a constant so $\vec{x}(t) = 0$. $\vec{x}_0$ is an equilibrium point, so

$$\vec{f}(\vec{x}(t)) = \vec{f}(\vec{x}_0) = 0.$$ Thus $\vec{x}(t) = \vec{f}(\vec{x}(t))$. \qed

2.3.1 Example

Recall the simple pendulum from the previous section.

$$\vec{f}\left(\begin{bmatrix} \theta \\ v \end{bmatrix}\right) = \begin{bmatrix} v \\ -\sin(\theta) \end{bmatrix}$$

We can solve for the equilibrium points with $\theta \in [-\pi, \pi]$ and we find $(\theta_0, v_0) \in \{(-\pi, 0), (0, 0), (\pi, 0)\}$. The initial condition $(0, 0)$ corresponds to the instance when the pendulum is hanging straight down. The initial conditions $(-\pi, 0), (\pi, 0)$ correspond to the instances when the pendulum is standing straight up and whether $\theta$ is considered positive or negative. If the pendulum starts at either of these positions with zero velocity and no forces acting on it besides gravity then it will stay at the starting position.

2.4 Linearization

The linearization of an ODE is used to see the qualitative behavior of an ODE in a very small neighborhood around a point. This is an ideal tool for analyzing what happens around equilibrium points.

Definition 6. The linearization of $\dot{\vec{x}} = \vec{f}(\vec{x})$ at the equilibrium point $\vec{x}_0$ is the linear ODE $\dot{\vec{y}} = D\vec{f}(\vec{x}_0) \cdot \vec{y}$.

Explicitly, this means $y : \mathbb{R} \to \mathbb{R}^n$ is a function,

$$\vec{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

is a vector field and

$$D\vec{f}(\vec{x}_0) \cdot \vec{y} = \begin{bmatrix} \frac{\partial f_1(\vec{x}_0)}{\partial y_1} & \frac{\partial f_1(\vec{x}_0)}{\partial y_2} & \cdots & \frac{\partial f_1(\vec{x}_0)}{\partial y_n} \\ \frac{\partial f_2(\vec{x}_0)}{\partial y_1} & \frac{\partial f_2(\vec{x}_0)}{\partial y_2} & \cdots & \frac{\partial f_2(\vec{x}_0)}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n(\vec{x}_0)}{\partial y_1} & \frac{\partial f_n(\vec{x}_0)}{\partial y_2} & \cdots & \frac{\partial f_n(\vec{x}_0)}{\partial y_n} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

5
2.4.1 Example

Now we will find the linearization of the equilibrium points for the pendulum example. Recall we have the system.

\[
\begin{bmatrix}
\dot{\theta} \\
\dot{v}
\end{bmatrix} = \begin{bmatrix}
v \\
-\sin(\theta)
\end{bmatrix} = \vec{f}(\theta, v) = \begin{bmatrix}
f_1(\theta, v) \\
f_2(\theta, v)
\end{bmatrix}
\]

Using the equation \( \vec{y} = D\vec{f}(x_0) \cdot \vec{y} \) we find that

\[
D\vec{f}(\theta, v) = \begin{bmatrix}
0 & 1 \\
-\cos(\theta) & 0
\end{bmatrix}
\]
so \( f(0,0) = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \).

This means that

\[
\vec{y} = \begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} = \begin{bmatrix}
y_2 \\
y_1
\end{bmatrix}.
\]

If we draw the phase plane portrait of the linearized system we would get something like Figure 2(b) near \((0,0)\). Now consider the other two equilibrium points \((\pi,0)\) and \((-\pi,0)\). Since our linearization matrix only depends on \(-\cos(\theta)\) and we know that \(\cos(\theta)\) is an even function, the linearization matrix for both points will be equal. So we have

\[
\vec{f}(\pm\pi,0) = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}.
\]

This means that

\[
\vec{y} = \begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} = \begin{bmatrix}
y_2 \\
y_1
\end{bmatrix}.
\]

If you draw the phase plane portrait of the linearized system you will get something like Figure 2(b) near \((\pm\pi,\pi)\).
2.5 Stability

As you probably noticed the linearization of the equilibrium points indicates drastically different behavior at \( \theta = 0 \) point compared to the behavior at the \( \theta = \pm \pi \) points. This different behavior is a key point when talking about stability of equilibrium points.

**Definition 7.** A solution \( \vec{x}(t) \) is **stable** if \( \forall \varepsilon > 0 \exists \delta > 0 \) such that if \( \vec{y}(t) \) is a solution with \( |\vec{x}(0) - \vec{y}(0)| < \delta \) then \( |\vec{x}(t) - \vec{y}(t)| < \varepsilon \forall t \geq 0 \). Otherwise the solution is **unstable**.

A quick look at the phase plane portraits of the linearization of the equilibrium points of the pendulum illustrates this very well. Notice that in the linearization of the \( \theta = 0 \) point near by solutions are ellipses and therefore stay near the equilibrium point for all time.

If you look at the phase plane portraits of linearization of the other two points, \( \theta = \pm \pi \) you see that if the initial condition is just a little away from the equilibrium point that it follows one of the hyperbolic paths which get farther away from the equilibrium point as time goes on. These hyperbolic paths translate into real life in how if the pendulum is just a little short of completely vertical it will fall back down then swing back and forth unless stopped by an external force. On the other hand, if the initial condition has the pendulum perfectly vertical but has even the smallest velocity of positive or negative value, then it will swing around and around without changing direction. Thus the \( \theta = \pm \pi \) equilibrium points are unstable points.

2.6 Integrals

**Definition 8.** A function \( H : \mathbb{R}^n \rightarrow \mathbb{R} \) is an **integral** if \( \frac{d}{dt}H(\vec{x}(t)) = 0 \) for all solutions \( \vec{x} \).

**2.6.1 Example**

We will now find the integral for the simple pendulum. Recall we have the differential equation

\[
\ddot{x} + \sin(x) = 0.
\]

Now we multiply the whole equation by \( \dot{x} \) so we have

\[
\dot{x}\ddot{x} + \dot{x}\sin(x) = 0.
\]

Which is equivalent to the following equation.

\[
\frac{d}{dt}(\frac{1}{2}\dot{x}^2 - \cos(x)) = 0
\]
Recall \( \dot{\theta} = \nu \), thus we have the integral
\[
H(\theta, \nu) = \frac{1}{2} \nu^2 - \cos(\theta).
\]

# 3 Orbits of Celestial Bodies

## 3.1 Newton’s Law of Universal Gravitation

### 3.1.1 The 2-body Problem

Given 2 Bodies, one at \( \vec{x}_1 \) with mass \( m_1 \) and the other at \( \vec{x}_2 \) with mass \( m_2 \), then the force exerted by \( m_2 \) on \( m_1 \) is
\[
\vec{F}_1 = G m_1 m_2 \frac{(\vec{x}_2 - \vec{x}_1)}{| \vec{x}_2 - \vec{x}_1 |^3}
\]
And the force exerted on \( m_2 \) by \( m_1 \) is
\[
\vec{F}_2 = G m_1 m_2 \frac{(\vec{x}_1 - \vec{x}_2)}{| \vec{x}_1 - \vec{x}_2 |^3}
\]
where \( G \) is the gravitational constant. Using Newton’s second law, \( F = ma \), we can rewrite these equations as a system of 2nd order differential equations:
\[
\begin{align*}
\ddot{x}_1 &= \frac{G m_1 m_2 (\vec{x}_2 - \vec{x}_1)}{| \vec{x}_2 - \vec{x}_1 |^3} \\
\ddot{x}_2 &= \frac{G m_1 m_2 (\vec{x}_1 - \vec{x}_2)}{| \vec{x}_1 - \vec{x}_2 |^3}
\end{align*}
\]

The fact that angular momentum is conserved allows us to rotate the coordinate system so that \( m_1 \) and \( m_2 \) only move on a plane and their positions can be described by real numbers \( x_1 \) and \( x_2 \) rather than vectors. By setting the center of mass at the origin and assuming that \( y \) represents the distance between the bodies such that \( y = x_1 - x_2 \) and \( M = G m_1 + G_2 \), we can express the problem as the differential equation
\[
\ddot{y} = -\frac{M}{y^2}
\]
Then we are able to find an implicit solution for \( y \):
\[
\frac{1}{\sqrt{2c_1}} \left[ \sqrt{y \left( y + \frac{M}{c_1} \right) - \frac{M}{c_1} \ln \left( \sqrt{y} + \sqrt{y + \frac{M}{c_1}} \right) \right] = -t + c_2
\]
where \( c_1 \) and \( c_2 \) are constants.
3.1.2 The n-Body Problem

Similarly, if there are three masses \( m_1, m_2, \) and \( m_3 \) with positions \( \vec{x}_1, \vec{x}_2, \) and \( \vec{x}_3 \) respectively, we derive the system:

\[
\begin{align*}
\ddot{x}_1 &= \frac{Gm_2(\vec{x}_2 - \vec{x}_1)}{|\vec{x}_2 - \vec{x}_1|^3} + \frac{Gm_3(\vec{x}_3 - \vec{x}_1)}{|\vec{x}_3 - \vec{x}_1|^3} \\
\ddot{x}_2 &= \frac{Gm_1(\vec{x}_1 - \vec{x}_2)}{|\vec{x}_1 - \vec{x}_2|^3} + \frac{Gm_3(\vec{x}_3 - \vec{x}_2)}{|\vec{x}_3 - \vec{x}_2|^3} \\
\ddot{x}_3 &= \frac{Gm_1(\vec{x}_1 - \vec{x}_3)}{|\vec{x}_1 - \vec{x}_3|^3} + \frac{Gm_2(\vec{x}_2 - \vec{x}_3)}{|\vec{x}_2 - \vec{x}_3|^3}
\end{align*}
\]

Given \( n \) masses \( m_1, m_2, \ldots, m_n \), the motion of the \( i \)th body is described by

\[
\ddot{x}_i = \sum_{i \neq j} Gm_j(\vec{x}_j - \vec{x}_i) / |\vec{x}_j - \vec{x}_i|^3
\]

There is no known solution for \( n \geq 3 \).

3.2 The Restricted 3-body Problem

3.2.1 Kepler’s Theorem

Kepler’s theorem applies to the 2-body problem. Recall that \( \vec{y}(t) = \vec{x}_1(t) - \vec{x}_2(t) \).

**Theorem 9.** Assume that \( \vec{y}(t) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \). If \( E = \frac{1}{2} |\vec{y}|^2 - \frac{M}{|\vec{y}|} < 0 \) and \( G_3 = y_1 \dot{y}_2 - y_2 \dot{y}_1 \neq 0 \) then \( \vec{y} \) traces out an ellipse with eccentricity \( b = (1 + 2EG_3^2)^{\frac{1}{2}} \), semimajor axis of length \( a = -\frac{1}{2E} \), and a focus at the origin. \( \vec{y}(t) \) sweeps out area at a constant speed and the period of the orbit is \( 2\pi a^{\frac{3}{2}} \).

**Remark 10.** If the orbit is a circle, then \( \vec{y}(t) \) rotates at a constant speed.

3.2.2 Rotated Coordinate System

The 3-body problem can be simplified by assuming that \( m_3 \) is much smaller than \( m_1 \) and \( m_2 \), so that the force it exerts on the other two masses is negligible. For example, if \( m_1 \) is the sun and \( m_2 \) is the earth, \( m_3 \) could be a satellite. The paths of \( m_1 \) and \( m_2 \) can be found by solving the 2-body problem and the motion of \( m_3 \) is described by the equation:

\[
\ddot{x}_3 = \frac{Gm_1(\vec{x}_1 - \vec{x}_3)}{|\vec{x}_1 - \vec{x}_3|^3} + \frac{Gm_2(\vec{x}_2 - \vec{x}_3)}{|\vec{x}_2 - \vec{x}_3|^3}
\]

If we assume that \( \vec{x}_1 \) and \( \vec{x}_2 \) have circular orbits, by Kepler’s Theorem, we can assume that they rotate at a constant speed. We can therefore
create a coordinate system that rotates at the same speed so that \( \vec{x}_1 \) and \( \vec{x}_2 \) always remain on the line \( y = 0 \) in the \( xy \)-plane. To do this, we first express the positions of \( m_1 \) and \( m_2 \) in polar coordinates:

\[
\vec{x}_1(t) = \begin{bmatrix} -r_1 \cos \left( \frac{2\pi t}{\tau} \right) \\ -r_1 \sin \left( \frac{2\pi t}{\tau} \right) \end{bmatrix}
\]

\[
\vec{x}_2(t) = \begin{bmatrix} \frac{m_1}{m_2} r_2 \cos \left( \frac{2\pi t}{\tau} \right) \\ \frac{m_1}{m_2} r_2 \sin \left( \frac{2\pi t}{\tau} \right) \end{bmatrix}
\]

where \( r_1 \) and \( r_2 \) are the distances of \( m_1 \) and \( m_2 \) from the origin respectively and \( \tau \) is the period of rotation. First we set the center of mass as the origin so that \( r_2 = \frac{m_1}{m_2} r_1 \). We will simply rename \( r = r_1 \).

We rotate the coordinates by \( -\frac{2\pi t}{\tau} \) using the matrix \( R \) to ensure that \( m_1 \) and \( m_2 \) always remain on the line \( y = 0 \):

\[
R(t) = \begin{bmatrix} \cos \left( \frac{2\pi t}{\tau} \right) & \sin \left( \frac{2\pi t}{\tau} \right) \\ -\sin \left( \frac{2\pi t}{\tau} \right) & \cos \left( \frac{2\pi t}{\tau} \right) \end{bmatrix}
\]

\[
\vec{x}'_1 = R \vec{x}_1 = \begin{bmatrix} -r \\ 0 \end{bmatrix}
\]

\[
\vec{x}'_2 = R \vec{x}_2 = \begin{bmatrix} \frac{m_1}{m_2} r \\ 0 \end{bmatrix}
\]

We then use \( R \) to find an equation for \( \vec{x}'_3 \):

\[
\vec{x}'_3 = R \vec{x}_3
\]

\[
\vec{x}'_3 = \dot{\vec{x}}_3 + \vec{x}'_3 = \ddot{\vec{x}}_3 + \vec{x}'_3 = \ddot{\vec{x}}_3 + 2 \dot{\vec{x}}_3 + R \vec{x}_3
\]

\[
= \begin{bmatrix} \frac{4\pi^2}{\tau^2} - \frac{G m_1}{r_{13}} - \frac{G m_2}{r_{23}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_3 + \frac{G m_1}{r_{13}} \vec{x}'_1 + \frac{G m_2}{r_{23}} \vec{x}'_2 
\end{bmatrix}
\]

where \( r_{13} \) is the distance between \( m_1 \) and \( m_3 \) and \( r_{23} \) is the distance between \( m_2 \) and \( m_3 \). Using this equation, we will create a system of first order differential equations with the values \( x, y, u, v \), such that \( x \) and \( y \) are the components of \( \vec{x}_3 \) and \( u \) and \( v \) are their respective velocities. We also substitute the components of \( \vec{x}_1 \) and \( \vec{x}_2 \).

\[
\begin{cases}
\dot{x} = u \\
\dot{y} = v \\
\dot{u} = 4\pi^2 \frac{x}{\tau^2} + (\frac{r_{13}}{r_{13}}) \frac{G m_1}{r_{13}} + \left( \frac{m_1}{m_2} r - x \right) \frac{G m_2}{r_{23}} + \frac{4\pi}{\tau} v \\
\dot{v} = 4\pi^2 \frac{y}{\tau^2} - \frac{G m_1}{r_{13}} - \frac{G m_2}{r_{23}} - \frac{4\pi}{\tau} u
\end{cases}
\]
We can further simplify the system by changing our units of time, mass, and distance so that $G = 1$, $m_1 + m_2 = 1$ and $r \left(1 + \frac{m_1}{m_2}\right) = 1$. Since $m_2 = 1 - m_1$, we will rename $m_1$ to simply be $\mu$. This makes the system becomes:

$$
\begin{align*}
\dot{x} &= u \\
\dot{y} &= v \\
\dot{u} &= x + (\mu - 1 - x) \frac{\mu}{r_{13}} + (\mu - x) \frac{1 - \mu}{r_{23}} + 2v \\
\dot{v} &= y - y \frac{\mu}{r_{13}} - y \frac{1 - \mu}{r_{23}} - 2u
\end{align*}
$$

where $r_{13} = \sqrt{(1 - \mu + x)^2 + y^2}$ and $r_{23} = \sqrt{(-\mu + x)^2 + y^2}$. Note that $r = 1 - \mu$.

### 3.2.3 Lagrangian Equilibrium Points

There are five equilibrium points in our system at which both $u$ and $v$ will be 0. Investigation of these points is essential to the project. We will refer to these as L1, L2, L3, L4, and L5. L1, L2, and L3 exist on the line connecting $m_1$ and $m_2$, which is the $x$-axis in the rotating coordinate system. L4 and L5 each form an equilateral triangle with $m_1$ and $m_2$ as the other two vertices. L4 and L5 are stable equilibrium points and L1, L2, and L3 are unstable. This will be revisited in section 5.2.

![Diagram of equilibrium points](image)

We solve for $L4$ and $L5$ geometrically to find that $x_{L4,L5} = \mu - \frac{1}{2}$ and $y_{L4,L5} = \pm \frac{\sqrt{3}}{2}$. We can easily prove that these are equilibrium points:
Proof. Since \( u,v=0 \) and \( r_{13} = r_{23} = 1 \)

\[
\dot{u} = \mu - \frac{1}{2} + \left[ \mu - 1 - \left( \mu - \frac{1}{2} \right) \right] \mu + \left[ \mu - \left( \mu - \frac{1}{2} \right) \right] (1 - \mu) + 2(0)
\]

\[
= \mu - \frac{1}{2} \mu - \frac{1}{2} \mu + \frac{1}{2}
\]

\[
= 0
\]

\[
\dot{v} = \pm \sqrt{3} \mu \pm \sqrt{3} \mu \pm \sqrt{3} (1 - \mu) - 2(0)
\]

\[
= \pm \sqrt{3} \mu \pm \sqrt{3} \mu \pm \sqrt{3} \mu
\]

\[
= 0
\]

To find \( L_1, L_2, \) and \( L_3 \), we must solve the three equations:

\[
0 = x L_1 + (\mu - 1 - x L_1) \frac{\mu}{(r + x L_1)^3} + (\mu - x L_1) \frac{1 - \mu}{(1 - r - x L_1)^3}
\]

\[
0 = x L_2 + (\mu - 1 - x L_2) \frac{\mu}{(r + x L_2)^3} + (\mu - x L_2) \frac{1 - \mu}{(-1 + r + x L_2)^3}
\]

\[
0 = x L_3 + (\mu - 1 - x L_3) \frac{\mu}{(-r - x L_3)^3} + (\mu - x L_3) \frac{1 - \mu}{(1 - r - x L_3)^3}
\]

### 3.2.4 Jacobi Integral

The Jacobi Integral is a function that produces the total energy per unit mass of the system, as it is the only known conserved quantity of the 3-body problem:

\[
E = \frac{1}{2}(\dot{u}^2 + \dot{v}^2) - \frac{1}{2}(x^2 + y^2) - \left( \frac{\mu}{\sqrt{(1 - \mu + x)^2 + y^2}} + \frac{1 - \mu}{\sqrt{(\mu + x)^2 + y^2}} \right)
\]

The first term represents the kinetic energy, as it is only dependent on \( u \) and \( v \) and the second two terms represent the potential energy as they are dependent on \( x, y \), and \( \mu \).

### 4 ODE45

In finding the numerical approximations of the solutions to the differential equations for both the pendulum problem and the restricted 3-body problem, we used the MATLAB function ode45. This ode45 solver
is based on an explicit Runge-Kutta (4,5) formula, the Dormand-
Prince pair. It is a one-step solver in computing \( y(t_n) \), it needs
only the solution at the immediately preceding time point, \( y(t_{n-1}) \).
This function takes three arguments. The first of these is the function
name of a function containing the system of first order differential
equations \texttt{ode45} will solve. The second is the vector containing the
initial time and the final time over which \texttt{ode45} will integrate the
system. The last argument is a vector of initial conditions. \texttt{ode45}
returns a column vector with the time at each instance for which the numerical
approximation is calculated and a matrix where each column contains
the approximated values of the solution of the ODE at the particular
time for a variable in your system of ODEs. In the programs we used
to numerically approximate solutions for the restricted 3-body prob-
lem we needed more precision. To achieve this, we used the \texttt{odeset}
function which created a structure \texttt{options} that allowed us to adjust
the relative error tolerance and the absolute error tolerance. The relative
error tolerance is relative to the whole matrix of solutions and
the absolute tolerance determines the threshold of acceptable error for
each individual component of the solution matrix. The default relative
error tolerance, 1e-3, corresponds to about 0.1% accuracy. The
default absolute error tolerance is 1e-6. In the program we used to
find approximations for the restricted 3-body problem, \texttt{solveL1.m} we
used \texttt{AbsTol} = 1e-10 and \texttt{RelTol} = 1e-7 [?].

4.1 Example

The following is an example of the code we used to solve the system
of differential equations of the pendulum. First we have the m-file
we called \texttt{system\_ex(t,x)} which defines the system of differential
equations

\[
\begin{bmatrix}
\dot{\theta} \\
\dot{v}
\end{bmatrix} = \begin{bmatrix}
v \\
-\sin(\theta)
\end{bmatrix}.
\]

If we compare this system to the following code, we see that \( \dot{\theta} = x(1) \),
\( \dot{v} = x(2) \), \( v = x(1) \), and \( \theta = x(2) \). Note that while the function
takes a time \( t \) it is not used in the system because the system does
not depend on time. The function \texttt{system\_ex} takes a time argument
only because the \texttt{ode45} function requires it.

\begin{verbatim}
function [xprime] = system_ex(t,x)
xprime(1,1) = -sin(x(2));
xprime(2,1) = x(1);
end
\end{verbatim}
This is the m-file we called `pendulums.m` that uses `ode45` to solve the system and plots the solutions for the initial conditions with $\theta = [-5, 5]$ with a step of .2 and zero velocity.

```matlab
hold on
for i=-5:.2:5
  [t,y] = ode45('system_ex',[0,11],[i 0]);
  x1=y(:,1);
  x2=y(:,2);
  plot(x2,x1);
  plot(-x2,x1);
end
hold off
```

Figure 3: Pendulum

Figure 3 is the figure plotted by this file in MATLAB.

5 Analysis of Linearization at Lagrange Points

5.1 Linearization at Lagrange Points

Another way to analyze the behavior of solutions near the equilibrium points is by studying the linearization of the system. Recall that the linearization of $\ddot{x} = Df(\vec{x}_0)\vec{y}$. Where

$$
\vec{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}
$$

and
\[ f_1(x, y, u, v) = u \]
\[ f_2(x, y, u, v) = v \]
\[ f_3(x, y, u, v) = x + (\mu - 1 - x) \frac{\mu}{r_{13}} + (\mu - x) \frac{1 - \mu}{r_{23}} + 2v \]
\[ f_4(x, y, u, v) = y - y \frac{\mu}{r_{13}} - y \frac{1 - \mu}{r_{23}} - 2u \]

The entries of the linearization matrix will be the partial derivatives of \( f_1, f_2, f_3, \) and \( f_4 \) in respect to \( x, y, u, \) and \( v \):

\[
\begin{bmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\
\frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \\
\frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial y} & \frac{\partial f_4}{\partial u} & \frac{\partial f_4}{\partial v}
\end{bmatrix}
\]

To find the linearization at \( L_4 \), we substitute the initial conditions, \( x = \mu - \frac{1}{2}, \ y = \sqrt{3} \), \( u = 0 \), and \( v = 0 \) to find the matrix:

\[
L_4 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\frac{3}{4} & \frac{6\sqrt{3}\mu-3\sqrt{3}}{4} & 0 & 2 \\
\frac{6\sqrt{3}\mu-3\sqrt{3}}{4} & \frac{9}{4} & -2 & 0
\end{bmatrix}
\]

By substituting \( x = \mu - \frac{1}{2} \) and \( y = -\frac{\sqrt{3}}{2} \), we get a similar linearization for \( L_5 \):

\[
L_5 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\frac{3}{4} & -\frac{6\sqrt{3}\mu+3\sqrt{3}}{4} & 0 & 2 \\
\frac{-6\sqrt{3}\mu+3\sqrt{3}}{4} & \frac{9}{4} & -2 & 0
\end{bmatrix}
\]

To obtain the matrices for \( L_1, L_2, \) and \( L_3 \) we would also substitute the coordinates of each Lagrangian point, but because they are roots of high degree polynomials, we are unable to solve for these points exactly, so we set \( \mu \) to be the mass of the sun (0.999996996) where \( 1 - \mu \) is the mass of the earth and found numerical solutions. We were then able to find the linearizations:

\[
L_1 = \begin{bmatrix}
0 & 0 & 1.0000 & 0 \\
0 & 0 & 0 & 1.0000 \\
9.1289 & 0 & 0 & 2.0000 \\
0 & -3.0644 & -2.0000 & 0
\end{bmatrix}
\]
To find a given vector $\vec{v}(t)$ in respect to a linearization, let’s say $L_4$, we compute $e^{Lt}\vec{v}$.

**5.2 Eigenvalues and Stability**

We can analyze the stability of the Lagrangian points by computing the eigenvalues and eigenvectors. However creating phase plane portraits is difficult because the system exists in $\mathbb{R}^4$, so we will only be able to produce plots in terms of two or three of the variables. To compute the eigenvalues of $L_4$, we must solve for $\lambda$ when $\det(L_4 - \lambda I) = 0$:

$$
\det(L_4 - \lambda I) = \lambda^4 + \lambda^2 + \frac{27\mu(1 - \mu)}{4} = 0
$$

Solving $\det(L_5 - \lambda I) = 0$ gives the same equation. The solutions to this equation are

$$
\lambda_1 = \sqrt{-\sqrt{27\mu^2 - 27\mu + 1} - 1} \\
\lambda_2 = -\sqrt{-\sqrt{27\mu^2 - 27\mu + 1} - 1} \\
\lambda_3 = \sqrt{\sqrt{27\mu^2 - 27\mu + 1} - 1} \\
\lambda_4 = -\sqrt{\sqrt{27\mu^2 - 27\mu + 1} - 1}
$$

Since, by construction, $0 < \mu < 1$, the eigenvalues are all strictly imaginary. Therefore, points nearby will orbit the equilibrium point. This is what we expect, since $L_4$ and $L_5$ are stable. When $\mu = 0.999999996$ as it is with the sun and the earth, the eigenvalues become...
\[ \lambda_1 = 0.99999i \]
\[ \lambda_2 = -0.999989i \]
\[ \lambda_3 = 0.00450i \]
\[ \lambda_4 = -0.00450i \]

To find the eigenvectors we solve \((L_4 - \lambda_n I) \vec{v}_n = \vec{0}\) for \(n = 1, 2, 3, 4\). Staying in the sun/earth system, the eigenvectors are

\[ \vec{v}_1 = \begin{bmatrix} 0.57009 \\ 0.22787 + 0.35082i \\ -0.00000 + 0.57009i \\ -0.35082 + 0.22787i \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0.57009 \\ 0.22787 - 0.35082i \\ -0.00000 - 0.57009i \\ -0.35082 - 0.22787i \end{bmatrix}, \]

\[ \vec{v}_3 = \begin{bmatrix} 0.86601 \\ 0.49999 + 0.00347i \\ -0.00000 + 0.00390i \\ -0.00002 + 0.00225i \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 0.86601 \\ 0.49999 - 0.00347i \\ -0.00000 - 0.00390i \\ -0.00002 - 0.00225i \end{bmatrix}. \]

We use the same process to compute the eigenvalues and eigenvectors of \(L_1\):

\[ \lambda_1 = -2.53400 \]
\[ \lambda_2 = 2.53400 \]
\[ \lambda_3 = 2.08727i \]
\[ \lambda_4 = -2.08727i \]

\[ \vec{v}_1 = \begin{bmatrix} 0.32377 \\ 0.17298 \\ -0.82043 \\ -0.43834 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -0.32377 \\ 0.17298 \\ -0.82043 \\ 0.43834 \end{bmatrix}, \]

\[ \vec{v}_3 = \begin{bmatrix} -0.12777 + 0.00000i \\ 0.00000 - 0.41274i \\ 0.00000 - 0.26668i \\ 0.86151 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} -0.12777 - 0.00000i \\ 0.00000 + 0.41274i \\ 0.00000 + 0.26668i \\ 0.86151 \end{bmatrix}. \]

\(L_1\) has two imaginary eigenvalues, so it will orbit around the equilibrium point, but because it also has a positive real eigenvalue, points will spiral outward. Therefore, \(L_1\) is not stable.
5.3 Change of Basis

Unfortunately plotting the linearizations is not possible because we cannot graph in 4-space. However, we can examine the projection onto various planes and spaces. We must change to a basis consisting of eigenvectors to best understand what is happening. Given four linearly independent vectors, \( \vec{b}_1, \vec{b}_2, \vec{b}_3, \) and \( \vec{b}_4, \) we create the matrix

\[
B = \begin{bmatrix}
\vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4
\end{bmatrix}
\]

To find a vector \( \vec{v} \) in respect to the new basis, we compute \( B^{-1} \vec{v} \). Recall that \( L_4 \) has four complex eigenvectors, where \( \vec{v}_1 \) and \( \vec{v}_4 \) is the conjugate of \( \vec{v}_3 \). Let us make the basis to be the real and imaginary parts of \( \vec{v}_1 \) and \( \vec{v}_3 \) so that we have

\[
B = \begin{bmatrix}
\Re(\vec{v}_1) & \Im(\vec{v}_1) & \Re(\vec{v}_3) & \Im(\vec{v}_3)
\end{bmatrix} = \begin{bmatrix}
0.57009 & 0 & 0.86601 & 0 \\
0.22787 & 0.35082 & 0.49999 & 0.00347 \\
-0.00000 & 0.57009 & -0.00000 & 0.00390 \\
-0.35082 & 0.22787 & -0.00002 & 0.00225
\end{bmatrix}
\]

Instead of simply computing \( e^{L_4t} \vec{v} \) to find \( \vec{v}(t) \) in respect to the linearization at \( L_4 \), we must find \( B^{-1} e^{L_4t} \vec{v} \). In contrast, \( L_1 \) has two real eigenvectors, \( \vec{v}_1 \) and \( \vec{v}_2 \), and two complex vectors of which \( \vec{v}_4 \) is the conjugate of \( \vec{v}_3 \). We will make our basis to be the two real vectors and the real and imaginary parts of \( \vec{v}_3 \):

\[
B = \begin{bmatrix}
\vec{v}_1 & \vec{v}_2 & \Re(\vec{v}_3) & \Im(\vec{v}_3)
\end{bmatrix} = \begin{bmatrix}
0.32377 & -0.32377 & -0.12777 & 0.00000 \\
0.17298 & 0.17298 & 0.00000 & -0.41274 \\
-0.82043 & -0.82043 & 0.00000 & -0.26668 \\
-0.43834 & 0.43834 & 0.86151 & 0
\end{bmatrix}
\]

As with \( L_1 \), vectors can be found in respect to \( B \) by computing \( e^{L_1t} \vec{v} \). We are now able to plot the projection of a given linearization onto either a plane or space in respect to two or three of the basis vectors.

5.4 Examples

5.4.1 Plots at the Linearization of \( L_4 \)

This is a plot of

\[
\begin{bmatrix}
0.0001 \\
0.0001 \\
0 \\
0
\end{bmatrix}
\]

in respect to the \( xy \)-plane over \( 0 < t < 1000 \). We expect the orbits to wind around the surface of a torus but that is hard to see in these coordinates.
This is the plot of \[
\begin{bmatrix}
0.0001 \\
0.0001 \\
0 \\
0
\end{bmatrix}
\]
in respect to the plane formed by \(\vec{v}_1\) and \(\vec{v}_2\) over \(0 < t < 1000\). The orbit forms a circle, as we expect, because it is the projection on to the plane of two complex eigenvectors. It represents winding around a torus in one direction.

This is a plot of \[
\begin{bmatrix}
0.0001 \\
0.0001 \\
0 \\
0
\end{bmatrix}
\]
in respect to the space formed by \(\vec{v}_1\), \(\vec{v}_2\), and \(\vec{v}_3\) over \(0 < t < 1000\).
5.4.2 Plots at the Linearization of L1

This is a plot of \[
\begin{pmatrix}
0.0001 \\
0.0001 \\
0 \\
0
\end{pmatrix}
\]
in respect to the hyperbolic plane formed by the real eigenvectors, \( v_1 \) and \( v_2 \).

This is a plot of \[
\begin{pmatrix}
.1 \\
0 \\
-3601655041499258; 0
\end{pmatrix}
\]
in respect to real vector \( v_1 \) and the real and imaginary part of \( v_3 \). Note that the point orbits L1 at first and then shoots off towards infinity.

This is a plot of the same as above, but in respect to the plane formed by \( \vec{v}_1 \) and \( \vec{v}_2 \).

This is a plot of \( \langle 0.0001, 0.0001, 0, 0 \rangle \) in respect to the space formed by \( \vec{v}_1, \vec{v}_3, \) and \( \vec{v}_4 \) over \( 0 < t < 10 \). Notice that the point initially orbits L1, but then shoots off towards infinity.
6 A Path from Earth’s Orbit to L4

Using ODE45 and the linearizations of the equilibrium points, we can easily analyze the orbits of various initial conditions in the hopes of finding paths between equilibrium points that minimize fuel usage. The Jacobi Integral will allow us to determine the change of energy required to make the desired journey.

6.0.3 Using L1 as a Slingshot to Get to L4

Let’s assume that we wished for a spacecraft beginning in earth’s orbit to go very close to L1 and then shoot off towards L4 where it will be captured and orbit or stay at the equilibrium point. Using the Jacobi Integral, we compute the energy at the initial conditions to be -1.500306675688968. By setting initial conditions that match this energy, we obtain trajectories that get in the vicinity of L4:
However, all of these orbits remain a certain distance away from L4, so we must adjust the energy of the initial conditions until we find a path that intersects L4. Doing so, we find that the initial condition $x=1.001040859733099$, $y=4.006220125610000e-004$, $u=-0.035468773164696$, and $v=0.064881573444297$, which does exactly that:

Another change in velocity is required to slow the spacecraft to come to rest at L4. By looking at the projection onto the hyperbolic plane formed by the real eigenvectors of the linearization at L1, we see how the unstable nature of this equilibrium point slingshots the spacecraft towards L4.

Using the Jacobi integral, we find that the change of energy per unit mass required to kick the spacecraft out of Earth’s orbit and then stop it at L4 is $2.803724847259878e-004$. We can compare this with a trajectory that does not use the instability of L1 to get to L4. A point starting with the initial conditions $x=1.001040859733099$, $y=4.006220125610000e-004$, $u=-0.030187384585606-1.074214289913$, and $v=.48$ produces the orbit:
The energy required for this orbit is 1.445991535916909, which is significantly higher than the path using the instability of L4.

7 Conclusion

The investigation was a success in that we accomplished our goal of using knowledge of equilibrium points to find a more cost effective route from earth to L4. Using L1 as a slingshot, we found a trajectory that required significantly less energy than one that went to L4 directly. However, there are opportunities for more research. There may exist an even more efficient route from L1 to L4 and investigation into such a route is worth pursuing. To cause the spacecraft to be captured by L4 we simply change the horizontal and vertical velocity to 0. We could attempt to find a way of slowing down the spacecraft a little before it reaches L1 enough to orbit L4 that requires less energy than the current method. Finally, it would be worthwhile to find trajectories that go between the other equilibrium points, such as L2 and L3, to allow for the reduction of cost of other missions. There are limitless trajectories to be investigated and taken advantage of.

References

