Traveling Wave Solutions with Hysteretic Behavior for a Forward-Backward Heat Equation

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Abstract
In this paper we find explicit solutions of traveling wave type for the equation $u_t = \Delta \alpha(u)$ where $\alpha$ is a polygonal with changing monotonicity. We see that for an unstable phase to exist it must be present in the initial data and that the long term the stable phase always invades the unstable phase. We also exhibit a solution of error function type that appears to be novel.[4].

1 Introduction
Equations of the type $u_t = \Delta \alpha(u)$ with $\alpha$ non-monotone have been studied since 1983 [11]. The first theoretical results about this problem took about 20 years to appear, mostly via viscosity approximations to the solution. This equation appears in population models of aggregating populations [9]. By the change of unknown $u = w_x$ it is also related to the equation $w_t = \left[ \phi(w_x) \right]_x$ (for a suitable phi) which arises in image reconstruction problems (where it is referred to as the Perona-Malik equation) and oceanography [10]. In [4] (section 7) the authors exhibit a series of explicit similarity solutions $x/\sqrt{t}$. In this paper we find explicit similarity solutions of traveling wave type $u(x,t) = v(x + bt)$ we also find a explicit similarity solution of form $u(x,t) = v(x/\sqrt{t})$ not found in [4]. We see that in all cases for traveling wave solutions to exist the stable phase (corresponding to branches on which $\alpha'$ is strictly positive) always invades the unstable phase; for an unstable phase to exist it must be present already in the initial data. There are no solutions of traveling wave type with two discontinuities.

\[ \alpha(u) = \begin{cases} 
  u - 1 & u > \frac{1}{2} \\
  -u & \frac{-1}{2} \leq u \leq \frac{1}{2} \\
  u + 1 & u < \frac{-1}{2} 
\end{cases} \]

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2 Traveling Wave Solutions

In this section, we will find solutions to $u_t = \alpha(u)_{xx}$ of the form $u(x, t) = v(x + bt)$. We will find discontinuous solutions, some will be two pieces of traveling waves that attach discontinuously with jump. The others will be a traveling wave on one half plane and a constant and the other. At the line of discontinuity, $\alpha(v)$ is continuous and the Rankine-Hugoniot condition are satisfied.

We began by using a traveling wave with $u(x, t) = v(x + bt) = v(\eta)$. From there we found:

$$u_t = v'(\eta)b$$
$$u_x = v'(\eta)$$
$$u_{xx} = v''(\eta)$$

Then setting $u_t = u_{xx}$ we found:

$$v'(\eta)b = v''(\eta)$$
$$b = \frac{v''(\eta)}{v'(\eta)}, v'(\eta) \neq 0$$

Integrate to find

$$b\eta + c = \log|v'(\eta)|$$
$$e^{(bn)}e^c = v'(\eta)$$
$$e^{(bn)}k_1 = v'(\eta)$$

and

$$\frac{1}{b} e^{(bn)}k_1 + k_2 = v(\eta)$$

In the unstable phase, $|u| < \frac{1}{2}$, we will seek $u$ such that both $u_t = 0$ and $\alpha(u)_{xx} = 0$. This
results in \( u = \text{constant} \) or \( u = c_1 x + c_0 \).

In what follows we will exhibit a solution that has (1) a solution that has a positive traveling wave and a constant branch, and (2) a solution that has a negative traveling wave branch and a constant branch. In fact, there are infinitely many solutions of either of this type if we allow the change from one stable phase to the other to happen for \( u > \frac{1}{2} \) and \( u < \frac{3}{2} \).

(1) Next we exhibit a solution with a branch that is a positive traveling wave and a branch that is a constant:

(1a) This solution attaches traveling wave and a constant both in stable phases: (jumping from \( u = \frac{1}{2} \) to \( u = \frac{-3}{2} \))

Using an increasing exponential in this case the jump condition reads as \( x'(t) = -\sqrt{2} (\frac{1}{2} - k_2) \). On the other hand, \( x'(t) = -\sqrt{2} \) thus, \( k_2 = \frac{-3}{2} \). Setting \( k_1 = 1 \), we have the following solution:

\[
\begin{align*}
  u(x, t) &= \begin{cases} 
    \frac{1}{\sqrt{2}} e^{\sqrt{2} x + 2t} - \frac{3}{2} & \text{for } x + \sqrt{2} t > \frac{\log(2\sqrt{2})}{\sqrt{2}} \\
    -\frac{3}{2} & \text{for } x + \sqrt{2} t < \frac{\log(2\sqrt{2})}{\sqrt{2}}
  \end{cases}
\end{align*}
\]

(1b) This solution attaches a traveling wave and a constant branch from the stable to the unstable phase. (jumping from \( u = 1 \) to \( u = 0 \)) The jump condition in this solution only affects the value of \( k_2 \).

\[
\begin{align*}
  u(x, t) &= \begin{cases} 
    \frac{1}{\sqrt{2}} e^{\sqrt{2} x + 2t} & \text{for } x + \sqrt{2} t > \frac{\log(2\sqrt{2})}{\sqrt{2}} \\
    0 & \text{for } x + \sqrt{2} t < \frac{\log(2\sqrt{2})}{\sqrt{2}}
  \end{cases}
\end{align*}
\]
(2) Similar solutions (with decreasing exponential branches) can be obtained using $b = -\sqrt{2}$ (jumping from $u = \frac{3}{2}$ to $u = \frac{-1}{2}$):

$$u(x, t) = \begin{cases} 
\frac{3}{2} & \text{for } x - \sqrt{2}t > \frac{-\log(2\sqrt{2})}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}}e^{-\sqrt{2}x + 2t} + \frac{3}{2} & \text{for } x - \sqrt{2}t < \frac{-\log(2\sqrt{2})}{\sqrt{2}}
\end{cases}$$

Notice that in all cases points $(x, t)$ initially in the unstable phase eventually lie in one of the stable phases. When the phase change occurs at either $u = \frac{-1}{2}$ or $u = \frac{1}{2}$ it is impossible to switch to values inside the unstable phase. The unstable phase occurs only when it is already present in the initial data. In the same way, there are no traveling wave solutions consisting of branches of exponentials and constants with two discontinuities.
3 Error Function Solutions

In [4] (section 7), the authors exhibit a number of explicit solutions with $\eta = \frac{x}{\sqrt{t}}$ as a similarity variable. Similarly the solution below, of this same type, appears to be new:

We began by finding

$$v(\eta)_t = -\frac{1}{2} v'(\eta) \eta \frac{1}{t}$$

and

$$v(\eta)_{xx} = \frac{1}{t} v''(\eta).$$

Setting the two equal, we find:

$$0 = v''(\eta) + \frac{1}{2} v'(\eta) \eta.$$

Integrating twice, we find the solution:

$$v(\eta) = -a \int_{\eta}^{\infty} e^{-z^2/4} dz + b.$$

We set $a = 1$ for the rest of our calculations since $a$ cancelled out and did not affect the jump condition.

Next we glue $u(x,t) = v\left(\frac{x}{\sqrt{t}}\right)$ to the constant $u = -\frac{1}{2}$, where $\eta = \frac{x}{\sqrt{t}}$.

Set $x'(\eta) = \frac{\eta}{2\sqrt{t}}$ and $v(\eta)_x = \frac{\eta - \eta^2}{\sqrt{t}}$ equal to each other using the jump condition from $u = \frac{3}{2}$ to $u = -\frac{1}{2}$ and found that there exists a critical point, $\eta$ s.t. $\eta_c = e^{-\frac{\eta^2}{4}}$. Thus we have a Free Boundary at $\eta_c \sqrt{t} = x$.

Setting $u(\eta_c) = -\int_{\eta}^{\infty} e^{-z^2/4} dz + b = \frac{3}{2}$ we find that $b = \frac{3}{2} + \int_{\eta}^{\infty} e^{-z^2/4} dz$. 
References


