A statistical mechanical model is defined on an oriented compact n-manifold equipped with a vertex ordered triangulation. State variables are located on edges and the analog of Boltzmann weights are assigned to top dimensional simplexes according to the Ansatz $\Pi f(\lambda(xy))^{\varepsilon_{xy}}$ where $\lambda$ is a complex valued function on the edge states. The powers $\varepsilon_{xy}$ depend only on the position of the edge within the ordered simplex and the combinatorial orientation of the simplex relative to the ambient orientation. Conditions are determined for which the resulting partition function is invariant under Pachner moves, thereby determining an invariant of PL manifolds. Instances of such invariants can differentiate between $S^3$, $S^2 \times S^1$ and the lens space $L(5,2)$.

Pachner Moves

- Want invariant to be independent of triangulation.
- Any triangulation of a $d$-dimensional manifold can be related by a finite sequence of bi-stellar flips.

3-dimensional Invariant Relation

- Receive a number for every labeling of a tetrahedron.
- $\alpha : X^6 \to K^*$ Here $X$ is the set of all labels.
- Want the value to be unchanged before and after the move. Thus $\alpha$ need satisfy $\alpha(\{abcd\})\alpha(\{acde\})^{-1}\alpha(\{bade\})\alpha(\{bcde\})\alpha(\{acde\})^{-1} = 1$

Labeling function

- $f : X \to K^*$
- Assign to each edge state an $n$th root of unity
- For a fixed labeling $\lambda$ take the Ansatz $\alpha(\lambda(\{abcd\})) = \prod_{x<y} f(\lambda(xy))^{\varepsilon_{xy}}$

3 Simplex Edge Orientation

<table>
<thead>
<tr>
<th>Orientation</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>ab</td>
<td>12 02 01 01 01</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ac</td>
<td>13 05 03 02 02</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ad</td>
<td>14 04 04 04 03</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>bc</td>
<td>23 23 13 12 12</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>bd</td>
<td>24 24 14 14 13</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>cd</td>
<td>34 34 34 24 23</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3 Simplex Edge Relations

$xy$ denotes $\varepsilon_{xy}$, the exponent on the edge in position $xy$.

<table>
<thead>
<tr>
<th>Edge Relation</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>ab-ab+ab</td>
<td>$ab = 0$</td>
</tr>
<tr>
<td>ab-ac+ac</td>
<td></td>
</tr>
<tr>
<td>ac-ac+ad+ad</td>
<td>$ad = 0$</td>
</tr>
<tr>
<td>ac+b-bc+bd</td>
<td>$f(\lambda)^{ac+b+bd} = 1$</td>
</tr>
<tr>
<td>ad+bd-bd</td>
<td></td>
</tr>
<tr>
<td>bc-bc+cd+cd</td>
<td>$cd = 0$</td>
</tr>
<tr>
<td>bd-bd-bd+cd</td>
<td></td>
</tr>
</tbody>
</table>

State Sum

- Let $A_T$ denote the set of all labellings of a triangulation. Then, the Ansatz together with
  $$\frac{1}{|A_T|} \sum_{\lambda \in A_T} \prod_{x<y} \alpha^\lambda(\lambda(\sigma))$$
  gives
  $\frac{1}{|A_T|} \sum_{\lambda \in A_T} \frac{1}{|F_{xy}|} \sum_{\sigma \in F_{xy}} f(\lambda(xy))^{\varepsilon_{xy}}$
- $|F_{xy}|$ is the number of edges $xy$

Applying the State Sum

- We assign an $n$th root of unity to each edge label and select $\varepsilon_{ac},\varepsilon_{bd}$ such that they sum to $m$.
- The total exponent on each edge label becomes a linear combination of $\varepsilon_{ac},\varepsilon_{bd}$ with coefficients of the signed frequency of showing up in the $ac,bc,bd$ positions respectively.

Example: $S^3$

- $S^3 = \partial \Delta^4$ Thus the product
  $$\alpha(\{abcd\})\alpha(\{acde\})^{-1}\alpha(\{bade\})\alpha(\{bcde\})\alpha(\{acde\})^{-1}$$
- Gives the State Sum for all labeling. The product is 1 for each labeling. Summing over all labels and dividing by $|A_T|$ gives 1.
- For every labeling set we get $Z(S^3) = 1$.

Example: $S^2 \times S^2$

- see Figures 3 and 4
- We got a number using a computer program
- With 4th roots of unity and labeling set 1 and 1, and $\varepsilon_{ab} = 1$, $\varepsilon_{cd} = 2$, $\varepsilon_{bd} = 1$ we found $Z(S^2 \times S^2) = 0$
- With 3rd roots of unity, labeling set 1, $\frac{1}{3} + i\frac{\sqrt{3}}{3}$ and $\varepsilon_{ab} = 1$, $\varepsilon_{cd} = 1$, $\varepsilon_{bd} = 1$ we found $Z(S^2 \times S^2) = \frac{27}{4}$

References


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