Characterization of Extinction Time and Approximation by Explicit Solutions for Total Variation Flow

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1D Case

**Proposition (HKS 2016)**

Let \(-\infty < x_1 < \cdots < x_m < \infty\), \(I_0 = (-\infty, x_1)\), \(I_i \in \{1, 2, \ldots, m-1\}\) \(I_i = (x_i, x_{i+1})\), and \(I_m = (x_m, \infty)\). Let \(A_0, \ldots, A_m \in \mathbb{R}\) and, for all \(i \in \{0, \ldots, m-1\}\), \(A_i \neq A_{i+1}\) and \(b_i = \text{sgn}(A_{i+1} - A_i)\). The solution to (1) with initial datum \(u_0 = \sum_{i=0}^{m-1} A_i \chi_{I_i}\) is given by

\[
u(x, t) = \sum_{i=1}^{m-1} \left[ b_i - b_{i-1} - A_i \right] \chi_{I_i}(x),
\]

(2)

until the least time, \(T\), where \(u(\cdot, \bar{T})|_{I_i} = \nu(\cdot, \bar{T})|_{I_i}\) for some \(i\). We then consolidate the intervals and iterate the proposition using \(\nu(x, \bar{T}) = u(x, \bar{T})\). We continue in this fashion until the solution stabilizes.

**Proposition (Regions of Constancy)**

Let \(u_1 \in L^1_{\text{loc}}(\mathbb{R}^n)\) be radial, \(u\) be the solution emanating from \(u_1\), and \(J \subset \mathbb{R}^n\) be an annulus or ball centered at \(x_0 = 0\). Then there is a sequence of explicit solutions \((u_k(x, t))_{k=1}^\infty\) such that \(u_k \xrightarrow{k \to \infty} u\) as \(k \to \infty\).

**Proposition (Local Behavior of Monotone Segments)**

Let \(u_1 \in L^1(\mathbb{R})\), \(J \subset \mathbb{R}\) be an open interval such that \(u_1(J) = \{c\}\) for some constant \(c \in \mathbb{R}\). Then there is a continuous function, \(c(t) : \mathbb{R} \to \mathbb{R}\), such that \(u(J(t)) = \{c(t)\}\).

**Extinction Time**

**Proposition**

Suppose \(u(x, t) \in L^1_{\text{loc}}(\mathbb{R}^n)\) is continuous with compact support. Let \(x_0\) denote the finite values of \(x\) such that \(u(x, 0) = 0\) and \(x \neq x_0\) such that for all \(x_n \in (x_n - \epsilon, x_n + \epsilon)\), \(x_n \neq x_0\), \(u(x_n, 0) \cdot u(x_n, t)\) for all \(x_n < x_n\), \(u(x_n) = 0\) and \(x_n > x_0\), \(u(x_n) = 0\). Next, let \(R_n = [x_n, x_{n+1}]\). Finally, denote \(M = \frac{1}{n} \max\{n, |u|dx, n_2, |u|dx, \ldots, n_n, |u|dx\}\.

Then, the extinction time of \(u, T\), is governed by the inequality

\[M \leq T \leq \frac{1}{2} \left( \left| (\int u^2) \right|, \left| (\int u) \right| \right).\]

**References**


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