Multi-Skein Invariants for Welded and Extended Welded Knots and Links

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Overview

- Background
  - Knots and Links
  - Reidemeister Moves
  - Invariants
- Setup
  - Yang-Type Skein Relation
- Results
  - Welded
  - Extended Welded
Knots

What is a mathematical knot?

- Tie a knot on a string and splice the ends together
- A **classical knot** is an embedding of the circle into $\mathbb{R}^3$ up to ambient isotopy
- A **link** is an embedding of a disjoint union of circles
- Useful to represent as a two-dimensional diagram
Diagram Examples\(^1\)

## Types of Crossings

<table>
<thead>
<tr>
<th>Feature</th>
<th>Diagram</th>
<th>4-Dimensional$^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positive Crossing</td>
<td><img src="positive_crossing.png" alt="Diagram" /></td>
<td><img src="positive_crossing_4D.png" alt="4D_Reidemeister_Moves" /></td>
</tr>
<tr>
<td>Negative Crossing</td>
<td><img src="negative_crossing.png" alt="Diagram" /></td>
<td><img src="negative_crossing_4D.png" alt="4D_Reidemeister_Moves" /></td>
</tr>
<tr>
<td>Virtual Crossing</td>
<td><img src="virtual_crossing.png" alt="Diagram" /></td>
<td><img src="virtual_crossing_4D.png" alt="4D_Reidemeister_Moves" /></td>
</tr>
<tr>
<td>Wen Mark</td>
<td><img src="wen_mark.png" alt="Diagram" /></td>
<td><img src="wen_mark_4D.png" alt="4D_Reidemeister_Moves" /></td>
</tr>
</tbody>
</table>

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Virtual, Welded, and Extended Welded Links

- Classical links can be extended by introducing these additional types of crossings
- **Virtual** and **Welded links** have cryptic geometric interpretations and are most conveniently understood diagrammatically
- **Extended welded links** encode restricted embeddings of tori in $\mathbb{R}^4$
Reidemeister Moves

Two link diagrams represent the same link if they are related by a sequence of Reidemeister moves:

- (R1a)
- (R1b)
- (R2)
- (R3)
- (V1)
- (V2)
- (V3)
- (M)
- (W)
- (E1)
- (E2)
- (E3)
- (E4)
Knot Invariants

• What is an invariant?
  • Quantitative method of telling knots apart
  • Equivalent knot diagrams compute to the same value
    • However, diagrams of different knots may also compute to the same value
  • If two knot diagrams compute to different values then they represent different knots
• Examples include the Alexander polynomial, the Jones polynomial, and the HOMFLY polynomial
  • Does not have to be a polynomial (e.g. tricolorability)
Yang’s Skein Relation

\[
\begin{align*}
[\underleftarrow{\underrightarrow{\cdot}}] &= a[\underleftarrow{\underrightarrow{\cdot}}] + b[\overleftarrow{\overrightarrow{\cdot}}] + c[\underleftarrow{\overrightarrow{\cdot}}] \\
[\underleftarrow{\overrightarrow{\cdot}}] &= x[\underleftarrow{\overrightarrow{\cdot}}] + y[\overleftarrow{\overrightarrow{\cdot}}] + z[\overleftarrow{\overrightarrow{\cdot}}]
\end{align*}
\]

- Semi-oriented, multi-skein relation
  - Positive and negative crossings replaced by virtual crossing, uncrossing, and turnaround
- \([L]\) means we compute the polynomial of the link \(L\) using the skein relation
- As stated, this skein relation does not compute a knot invariant
Goal

- Use Yang’s multi-skein relation to construct polynomial invariants of welded and extended welded links by imposing invariance under Reidemeister moves
Important Conventions

- Removing a disjoint unknot
  - \([L \cup \bigcirc] = t[L]\)
- V1
  - \(\begin{array}{c}
  L
  
  \end{array} = r \begin{array}{c}
  L
  \end{array}\)
  - \(r = \pm 1\)
- Trivially have invariance under V2 and V3; invariance under M follows directly from these
- With the skein relation and these conventions all knots reduce to \(P[\ ]\)
\[
\begin{align*}
\begin{bmatrix}
\end{bmatrix}^{(R2)} & = a \begin{bmatrix}
\end{bmatrix} + b \begin{bmatrix}
\end{bmatrix} + c \begin{bmatrix}
\end{bmatrix} \\
\begin{bmatrix}
\end{bmatrix} & = x \begin{bmatrix}
\end{bmatrix} + y \begin{bmatrix}
\end{bmatrix} + z \begin{bmatrix}
\end{bmatrix}
\end{align*}
\]
\[
\begin{align*}
\begin{pmatrix}
\text{R2} \\
\end{pmatrix}
&= x \begin{pmatrix}
\text{R2}
\end{pmatrix} + y \begin{pmatrix}
\text{ExtR2}
\end{pmatrix} + z \begin{pmatrix}
\text{ExtExtR2}
\end{pmatrix} \\
&= ax \begin{pmatrix}
\text{R2}
\end{pmatrix} + bx \begin{pmatrix}
\text{ExtR2}
\end{pmatrix} + cx \begin{pmatrix}
\text{ExtExtR2}
\end{pmatrix} + \\
&\quad ay \begin{pmatrix}
\text{R2}
\end{pmatrix} + by \begin{pmatrix}
\text{ExtR2}
\end{pmatrix} + cy \begin{pmatrix}
\text{ExtExtR2}
\end{pmatrix} + \\
&\quad az \begin{pmatrix}
\text{R2}
\end{pmatrix} + bz \begin{pmatrix}
\text{ExtR2}
\end{pmatrix} + cz \begin{pmatrix}
\text{ExtExtR2}
\end{pmatrix}
\end{align*}
\]
R2 Conditions

\[
\begin{bmatrix}
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\delta
\end{pmatrix}
\end{bmatrix}
= (ay + bx) \begin{bmatrix}
\times
\end{bmatrix} + (ax + by) \begin{bmatrix}
\times \times
\end{bmatrix}
+ (azr + cxr + bz + cy + czt) \begin{bmatrix}
\times
\end{bmatrix}
\]

- \( ay + bx = 0 \)
- \( ax + by = 1 \)
- \( azr + cxr + bz + cy + czt = 0 \)
We then derive the following equations for \( x, y, \) and \( z \):

\[
\begin{align*}
x &= \frac{-a}{b^2 - a^2} \\
y &= \frac{b}{b^2 - a^2} \\
z &= \frac{rac - bc}{(b^2 - a^2)(ar + b + ct)}
\end{align*}
\]

[ \( L \) ] is now invariant under R2
Welded

\[
\begin{align*}
\begin{bmatrix}
\begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw[->] (0,0) -- (1,1);
\draw[<->,dotted] (1,0) -- (0,1);
\end{tikzpicture}
\end{array}
\end{bmatrix}
&= a^2 \begin{bmatrix}
\begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw[->] (0,0) -- (1,1);
\draw[<->,dotted] (1,0) -- (0,1);
\end{tikzpicture}
\end{array}
\end{bmatrix} + ab \begin{bmatrix}
\begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw[->] (0,0) -- (1,0);
\draw[<->,dotted] (1,1) -- (0,1);
\end{tikzpicture}
\end{array}
\end{bmatrix} + ac \begin{bmatrix}
\begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw[->] (0,0) -- (1,1);
\draw[<->,dotted] (1,0) -- (0,1);
\draw[->] (0,0) -- (0,1);
\end{tikzpicture}
\end{array}
\end{bmatrix} + \\
abla \begin{bmatrix}
\begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw[->] (0,0) -- (1,1);
\draw[<->,dotted] (1,0) -- (0,1);
\end{tikzpicture}
\end{array}
\end{bmatrix} + \nabla \nabla \begin{bmatrix}
\begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw[->] (0,0) -- (1,1);
\draw[<->,dotted] (1,0) -- (0,1);
\draw[->] (0,0) -- (0,1);
\end{tikzpicture}
\end{array}
\end{bmatrix}
+ b^2 \begin{bmatrix}
\begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\end{tikzpicture}
\end{array}
\end{bmatrix} + bc \begin{bmatrix}
\begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw[->] (0,0) -- (1,1);
\draw[<->,dotted] (1,0) -- (0,1);
\end{tikzpicture}
\end{array}
\end{bmatrix} + c^2 \begin{bmatrix}
\begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\end{tikzpicture}
\end{array}
\end{bmatrix} + \\
\end{align*}
\]

\begin{flushright}
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\end{flushright}
\[
\begin{align*}
&= a^2 \begin{bmatrix} \text{Diagram 1} \end{bmatrix} + ab \begin{bmatrix} \text{Diagram 2} \end{bmatrix} + ac \begin{bmatrix} \text{Diagram 3} \end{bmatrix} + \\
&+ ab \begin{bmatrix} \text{Diagram 4} \end{bmatrix} + b^2 \begin{bmatrix} \text{Diagram 5} \end{bmatrix} + bc \begin{bmatrix} \text{Diagram 6} \end{bmatrix} + \\
&+ ac \begin{bmatrix} \text{Diagram 7} \end{bmatrix} + bcr \begin{bmatrix} \text{Diagram 8} \end{bmatrix} + c^2 \begin{bmatrix} \text{Diagram 9} \end{bmatrix} \end{align*}
\]
Welded Conditions

\[ b^2 \left[ \begin{array}{c} \includegraphics[height=1cm]{diagram1} \\ \includegraphics[height=1cm]{diagram2} \end{array} \right] + bc \left[ \begin{array}{c} \includegraphics[height=1cm]{diagram3} \\ \includegraphics[height=1cm]{diagram4} \end{array} \right] + bcr \left[ \begin{array}{c} \includegraphics[height=1cm]{diagram5} \\ \includegraphics[height=1cm]{diagram6} \end{array} \right] + c^2 \left[ \begin{array}{c} \includegraphics[height=1cm]{diagram7} \\ \includegraphics[height=1cm]{diagram8} \end{array} \right] \]

- This equation is only satisfied when we set \( b = 0 \) and \( c = 0 \)
- Instead we consider the diagrams in a more global context by considering closures
- We don’t actually need the sum of tangles to be equal
  - We only need the sum to be equal for all possible knots
All Possible Knots in 15 Closures

- Luckily, all possible knots reduce to only fifteen cases
  - Classical crossings on the outside reduce to either virtual crossings or no crossing by applying the skein relation
  - Any combination of virtual crossings on the outside can be reduced by V1, V2, and V3

- Consider all ways to connect the nodes without connecting any node to itself
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Result of Considering Closures

\[
b^2 \left( \begin{array}{c} x \\ y \\ z \end{array} \right) + bc \left( \begin{array}{c} a \\ b \\ c \end{array} \right) + bcr \left( \begin{array}{c} d \\ e \\ f \end{array} \right) + c^2 \left( \begin{array}{c} g \\ h \\ i \end{array} \right) = b^2 \left( \begin{array}{c} j \\ k \\ l \end{array} \right) + bc \left( \begin{array}{c} m \\ n \\ o \end{array} \right) + bcr \left( \begin{array}{c} p \\ q \\ r \end{array} \right) + c^2 \left( \begin{array}{c} s \\ t \\ u \end{array} \right)
\]

- Close each tangle in the above equation into a knot
- Three distinct equations not immediately satisfied
  1. \( b^2 + bc + bct + c^2 = b^2 t + bct^2 + bc + c^2 t \)
  2. \( b^2 + 2bct + c^2 t^2 = b^2 t + 2bc + c^2 \)
  3. \( b^2 t^2 + 2bct + c^2 = b^2 + 2bc + c^2 t \)
- \( t = 1 \) makes the trivial invariant
- \( b = \pm c; \ t = \mp 2 \)
Review of R2 and W

\[
\begin{align*}
\begin{bmatrix}
\downarrow & \downarrow \\
\end{bmatrix} &= a \begin{bmatrix}
\downarrow & \downarrow \\
\end{bmatrix} + b \begin{bmatrix}
\downarrow & \uparrow \\
\end{bmatrix} + c \begin{bmatrix}
\downarrow & \downarrow \\
\end{bmatrix} \\
\begin{bmatrix}
\downarrow & \downarrow \\
\end{bmatrix} &= x \begin{bmatrix}
\downarrow & \downarrow \\
\end{bmatrix} + y \begin{bmatrix}
\downarrow & \uparrow \\
\end{bmatrix} + z \begin{bmatrix}
\downarrow & \downarrow \\
\end{bmatrix}
\end{align*}
\]

- These relations yield values invariant under R2 when subject to the following conditions:

\[
\begin{align*}
x &= \frac{-a}{b^2 - a^2} \\
y &= \frac{b}{b^2 - a^2} \\
z &= \frac{rac - bc}{(b^2 - a^2)(ar + b + ct)}
\end{align*}
\]
Review of R2 and W

- If in addition either of the following conditions hold
  - \( b = c, t = -2 \) with \( r = \pm 1 \)
  - \( b = -c, t = 2 \) with \( r = \pm 1 \)

  then \( [L] \) becomes invariant under R2 and W

- Invariance under R3 follows directly from R2 and W
Writhe

\[ w(L) = \# \text{positive crossings} - \# \text{negative crossings} \]

\[ v(L) = \# \text{virtual crossings} \]


Then \( Y(L) = r^v(L)A^{-w(L)}[L] \) is an invariant of welded links.

- We have shown that \([L]\) is invariant under every Reidemeister move for welded links except for R1
- \( r^v(L)A^{-w(L)} \) is also invariant under these moves; hence \( Y(L) \) is as well
- All that remains is to show that \( Y(L) \) is invariant under R1 and V1
Invariance under R1 and V1

Let \( L = \text{[square]} \) and \( L' = \text{[square]}. \)

Then \( w(L') = w(L) + 1 \), and we have

\[
Y(L') = A^{-w(L')} [L'] \\
= A^{-(w(L)+1)} [L'] \\
= A^{-(w(L)+1)} A [L] \\
= A^{-w(L)} [L] \\
= Y(L).
\]

V1 and R1b are shown similarly.
R1 Via Writhe Correction

\[
\begin{align*}
\begin{bmatrix}
  L \\
\end{bmatrix} &= A \begin{bmatrix}
  L \\
\end{bmatrix} = (ar + bt + c) \begin{bmatrix}
  L \\
\end{bmatrix} \\
\begin{bmatrix}
  L \\
\end{bmatrix} &= A^{-1} \begin{bmatrix}
  L \\
\end{bmatrix} = (xr + yt + z) \begin{bmatrix}
  L \\
\end{bmatrix}
\end{align*}
\]

- Condition required for writhe correction:
  \((ar + bt + c)(xr + yt + z) = 1\)
- All 4 cases invariant under \(W\) satisfy this equation without additional constraints on the variables
- \(Y(L)\) is an invariant of welded knots and links
  \(Y(L) = r^{v(L)}(ar + bt + c)^{-w(L)}[ L ]\)
Extended Moves

- Need to do welded Reidemeister move with wen marks
- Note that E4 turns a positive crossing into a negative crossing
  - Writhe only works if $A = A^{-1}$
- We have invariance under E1 and E2 trivially
- Obtain invariance under E3 and E4 by considering closures
  - Must also allow wen marks to appear on the closure
  - E3 does not require any additional constraints
E4 Constraints

- We get the following equations from E4:

\[ ar + b + ct = xr + y + zt \]
\[ at + br + cr = xt + yr + zr \]
\[ ar + bt + c = xr + yt + z \]

- \( b = -c \Rightarrow a = 0 \) or \( b = c = 0 \)
- \( b = c \)
  - \( b = ra - 1 \)
  - \( b = ra + 1 \)
  - These give 1 or \(-1\), respectively, for writhe factor
The rational function \( Y(L) = r^{v(L)}(ar + bt + c)^{-w(L)}[L] \) is an invariant of welded knots and links, where \([L]\) is computed by the following skein relation

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{crossing}}
\end{array}
\end{array}
\end{align*}
= a \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{positive_crossing}}
\end{array}
\end{array} + b \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{virtual_crossing}}
\end{array}
\end{array} + c \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{negative_crossing}}
\end{array}
\end{array}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{crossing}}
\end{array}
\end{array}
\end{align*}
= x \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{positive_crossing}}
\end{array}
\end{array} + y \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{virtual_crossing}}
\end{array}
\end{array} + z \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{negative_crossing}}
\end{array}
\end{array}
\]

and \( v(L) \) and \( w(L) \) are given by

\[
\begin{align*}
w(L) &= \# \text{positive crossings} - \# \text{negative crossings} \\
v(L) &= \# \text{virtual crossings}
\end{align*}
\]
Summary

- Subject to all of the following relations:
  - \( x = \frac{-a}{b^2 - a^2} \)
  - \( y = \frac{b}{b^2 - a^2} \)
  - \( z = \frac{rac - bc}{(b^2 - a^2)(ct + ra + b)} \)
  - \( r = \pm 1 \)

- And subject to one of the following:
  - \( b = c, t = -2 \)
  - \( b = -c, t = 2 \)

- Moreover, if we take the \( b = c \) option above and impose one of the following, then \( Y(L) \) is an invariant of extended welded knots and links:
  - \( b = ra - 1 \)
  - \( b = ra + 1 \)
Example Calculations of Polynomial

- Here are some example calculations of our invariant, with the following conditions:
  - $r = 1$
  - $t = 2$
  - $c = -b$

\[
Y(\text{Diagram 1}) = \frac{4(a^2+b^2)}{(a+b)^2}
\]

\[
Y(\text{Diagram 2}) = \frac{4a}{a+b}
\]

\[
Y(\text{Diagram 3}) = 2
\]
Future Research

- Restrict our polynomial to classical knots and links
  - Vasseliev Invariants
- Computation of polynomial for different knot examples
- Solving for turnarounds to produce a HOMFLY type skein relation
- Obtain an invariant of ribbon torus-links by considering mirror imaging
- Explore other skein relations in the context of welded and extended welded knots and links

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Questions?

Please feel free to contact me at tal8458@truman.edu with further inquiries!

Additionally, a preprint of our paper is available on arXiv under the title “Multi-Skein Invariants for Welded and Extended Welded Knots and Links” (arXiv:1809.05874 [math.GT])