Introduction

We present a new class of invariants for welded and extended welded knots and links using a multi-skein relation, following Z. Yang’s approach for virtual knots. Using this skein-theoretic approach, we find sufficient conditions on the coefficients to obtain invariance under the classical and extended Reidemeister moves.

Background

The recently developed theory of welded and extended welded knots is a generalization of classical knot theory. While classical knot diagrams encode embeddings of the circle into $\mathbb{R}^3$, extended welded knot diagrams encode ribbon torus-links, a restricted class of embeddings of tori in $\mathbb{R}^3$.

A knot invariant is a simpler way of quantitatively distinguishing between knots and has the same value for equivalent knots. Diagrams of different knots may sometimes compute to the same value; however, if two knot diagrams compute to different values, then they necessarily represent different knots.

Yang’s Skein Relation:

\[
\begin{align*}
\begin{bmatrix} y \end{bmatrix} &= a \begin{bmatrix} x \end{bmatrix} + b \begin{bmatrix} y \end{bmatrix} + c \begin{bmatrix} z \end{bmatrix} \\
\begin{bmatrix} z \end{bmatrix} &= x \begin{bmatrix} y \end{bmatrix} + y \begin{bmatrix} z \end{bmatrix} + z \begin{bmatrix} x \end{bmatrix}
\end{align*}
\]

To remove an unknot component or a virtual crossing on one strand, we use the following:

\[
L \cup \begin{bmatrix} \varepsilon \end{bmatrix} = r \begin{bmatrix} L \end{bmatrix}
\]

We also require $r = \pm 1$.

In some cases, it is necessary to consider the writhe of a knot, $w(L)$:

\[
w(L) = \# \text{pos. crossings} - \# \text{neg. crossings}
\]

And in the case of knots with virtual crossings:

\[
v(L) = \# \text{virtual crossings}
\]

Two knot diagrams represent the same link if they are related under a series of Reidemeister moves. As stated, the skein relation is not an invariant; we must impose restrictions on the relations under these moves. For our invariant, we trivially have invariance under V2 and V3; invariance under M thus follows from these.

Reideemeister Moves

The first three rows make up the welded moves, and the addition of the last row makes up the extended welded moves.

Invariance under R2

Applying the skein relation to $R_2$, we get the following equation:

\[
\begin{bmatrix} y \end{bmatrix} = (ay + bx) \begin{bmatrix} x \end{bmatrix} + (az + by) \begin{bmatrix} y \end{bmatrix} + (az + cx + bx + cy + cz) \begin{bmatrix} z \end{bmatrix}
\]

We set the coefficients to the following values:

\[
\begin{align*}
ay + bx &= 0, \\
az + by &= 1, \\
ax + cx + bx + cy + cz &= 0,
\end{align*}
\]

and derive the following relations for $x$, $y$, and $z$:

\[
\begin{align*}
x &= \frac{y - b}{a} \\
y &= \frac{b}{y^2 - a^2} \\
z &= \frac{r \alpha c - bc}{(y^2 - a^2)(ar + b + ct)}
\end{align*}
\]

Extended Welded Moves

To represent removing a wenz mark we use the following:

\[
\begin{bmatrix} L \end{bmatrix} \cup \begin{bmatrix} \varepsilon \end{bmatrix} = s \begin{bmatrix} L \end{bmatrix}
\]

Invariance under R1 and W1.

Welded Move

Applying our skein relation to the welded move produces the following equation:

\[
\begin{align*}
\begin{bmatrix} y \end{bmatrix} &= (ay + bx) \begin{bmatrix} x \end{bmatrix} + (az + by) \begin{bmatrix} y \end{bmatrix} + (az + cx + bx + cy + cz) \begin{bmatrix} z \end{bmatrix} \\
\begin{bmatrix} z \end{bmatrix} &= (x^2 + yz + x + y) \begin{bmatrix} z \end{bmatrix} + (x^2 + yz + x + y) \begin{bmatrix} x \end{bmatrix}
\end{align*}
\]

Because the tangles within the bracket do not match, we consider all possible closures on three strands; this reduces down to 15 cases by the skein relation and the use of the virtual moves.

Results

When looking at the equations created from each case, only three are not immediately satisfied.

\[
\begin{align*}
b^2 + bc + be + c^2 &= b^2t + be + b + c^2t \\
t^2 + 2bc + c^2t &= b^2t + 2bc + c^2t \\
b^2t^2 + 2bc + c^2t &= b^2t + 2bc + c^2t
\end{align*}
\]

The welded move produces these possible values for $b, t$, and $v$:

\[
\begin{align*}
t &= 1, \quad \text{trivial case} \\
b &= c, \quad t = -2 \text{ with } r = \pm 1 \\
b &= -c, \quad t = 2 \text{ with } r = \pm 1.
\end{align*}
\]

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References
