Characterization of Extinction Time and Approximation by Explicit Solutions for Total Variation Flow

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Introduction
Total Variation Flow (TVF)

The TVF is given by $u_t = \text{div}_x \frac{\nabla_x u}{|\nabla_x u|}$.

- Previously studied by Andreu-Vaillo, Caselles, and Mazón; Bellettini, Caselles, and Novaga; Bonforte and Figalli
- Applications for TVF include image processing, preventative policing
Classical problem:
If \( u_I \in L^1_{loc}(\mathbb{R}^N) \) then a classical solution to the TVF is a differentiable function \( u : \mathbb{R}^N \times (0, T) \to \mathbb{R} \) with

\[
\frac{\partial u}{\partial t} = \text{div}_x \left( \frac{\nabla_x u}{|\nabla_x u|} \right).
\]
Total Variation Flow
Classical and Strong Solutions

- Classical problem:
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  \]

- Strong formulation with solution \( u : (0, T) \times \mathbb{R}^N \to \mathbb{R} \):
  \[
  \begin{cases}
    u_t(x, t) = \text{div}_x z(x, t) \text{ in } D'((\mathbb{R}^N) \times (0, \infty)) \\
    |z| \leq 1 \\
    \int_0^T \int_{\mathbb{R}^N} \nabla_x u \cdot z \, dxdt = \int_0^T \int_{\mathbb{R}^N} |\nabla_x u| \, dxdt
  \end{cases}
  \]
Definition (1/2)

Let \( P = \{ p \in W^{1,\infty}(\mathbb{R}) : p' \geq 0, \text{supp } p' \text{ is compact} \} \). For all \( \ell \in \mathbb{R} \), all \( \eta \in C^\infty((0, T) \times \mathbb{R}^N) \), with \( \eta \geq 0 \), \( \eta(t, x) = \phi(t)\psi(x) \), where \( \phi \in C^\infty_0((0, T)) \), \( \psi \in C^\infty_0(\mathbb{R}^N) \), and \( p \in P \), where \( j(r) = \int_0^r p(s) \, ds \).

\( u \in C([0, T]; L^1_{loc}) \) is an entropy solution if

- \( u(\cdot, t) \to u_1(\cdot) \) in \( L^1_{loc} \) as \( t \to 0^+ \).
- \( p(u) \in L^1_w((0, T); BV_{loc}(\mathbb{R}^N)) \) for all \( p \in P \).
Definition (2/2)

- there exists \( z \in L^\infty(\mathbb{R}^N \times (0, T)) \) such that \( |z| \leq 1 \),

\[
    u_t = \text{div}_x z \text{ in } D'(\mathbb{R}^N \times (0, T))
\]

and

\[
    - \int_0^T \int_{\mathbb{R}^N} j(u - \ell) \eta_t + \int_0^T \int_{\mathbb{R}^N} \eta \, d|D(p(u - \ell))| \\
    + \int_0^T \int_{\mathbb{R}^N} z \nabla \eta p(u - \ell) \leq 0
\]
Entropy solutions meet a number of properties:

Theorem (BCN 2002)

For every $u_I \in L^1_{\text{loc}}$ there exists a unique entropy solution.
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**Theorem (BCN 2002)**

If $u_I, u_{I,n} \in L^1_{loc}(\mathbb{R}^N)$ with $u_{I,n} \rightarrow u_I$ in $L^1_{loc}$ have entropy solutions $u$ and $u_n$ respectively, then for all compact $K$,

$$\sup_{t \in (0,T)} \int_K |u(x, t) - u_n(x, t)| \, dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$
Approximations by Explicit Solutions

Main Result

Proposition (HKS 2016)

Let $A_1, \ldots, A_{m+1}$ be real constants with $A_{m+1} = 0$ and $A_i \neq A_{i+1}$ for $i = 1, \ldots, m$, and $0 = r_0 < r_1 < \cdots < r_m$. For $i = 1, \cdots, m$, let $\chi_i$ be the annulus with inner radius $r_{i-1}$ and outer radius $r_m$. Let $b_i = \text{sgn}(A_{i+1} - A_i)$ and $b_0 = 0$. If

$$u_I(x) = \sum_{i=1}^{m} A_i \chi_i(x).$$

Then the evolution of $u_I$ under TVF is given by

$$u(x, t) = \sum_{i=1}^{m} \left( \frac{b_i \mathcal{H}^{N-1}(\partial B_{r_i}) - b_{i-1} \mathcal{H}^{N-1}(\partial B_{r_{i-1}})}{\mathcal{L}^N(B_{r_i}) - \mathcal{L}^N(B_{r_{i-1}})} t + A_i \right) \chi_i(x).$$
Approximations by Explicit Solutions

Pictorial Example
Approximations by Explicit Solutions

Interpretations

- All points on an annuli move together
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Approximations by Explicit Solutions

Interpretations

- All points on an annuli move together
- Iterative process, we repeat every time two annuli merge
- Every annulus moves “independently"
- An annulus moves in the direction of its outer annulus neighbor
- Eventually, all annuli have zero height (time it takes = Extinction Time)
Approximations by Explicit Solutions

Pictorial Example

\[ u_I \]

\[ \Omega_1 \]

\[ \Omega_2 \]

\[ \Omega_3 \]

\[ A_1 \]

\[ A_2 \]

\[ A_3 \]
The outermost region need not have 0 height

Corollary (KLMV 2018)

If \( u_i(x) = \sum_{i=1}^{m+1} A_i \chi_i(x) \) (recall \( \Omega_{m+1} \) refers to the region out of the annuli), then

\[
u(x, t) = \sum_{i=1}^{m} \left( \frac{b_i \mathcal{H}^{N-1}(\partial B_{r_i}) - b_{i-1} \mathcal{H}^{N-1}(\partial B_{r_{i-1}})}{\mathcal{L}^N(B_{r_i}) - \mathcal{L}^N(B_{r_{i-1}})} \right) t + A_i \chi_i(x) + A_{m+1} \chi_{m+1}(x)
\]

and \( u(x, t) \to A_{m+1} \) as \( t \to \infty \).
Approximations by Explicit Solutions
Another Pictorial Example
Proposition (KLMV 2018)

Let $0 \leq u_I : \mathbb{R}^n \to \mathbb{R}$ be a continuous compactly supported radial initial datum with profile curve $f$ and let $u$ be the evolution of $u_I$ under TVF. Then there exists an increasing sequence of explicit solutions $u_n(x, t)$ such

$$u_n \xrightarrow{L^1_{loc}} u \text{ as } n \to \infty.$$
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$$ u_n \xrightarrow{L^1_{\text{loc}}} u \text{ as } n \to \infty. $$

Theorem (KLMV 2018)

Let $0 \leq u_I \in L^1_{\text{loc}}(\mathbb{R}^n)$ be radial initial data, $u$ be the evolution of $u_I$ under TVF, and $f(|x|) := u_I(x)$. Then there exists a sequence of explicit solutions $(u_k(x, t))_k$ such

$$ u_k \xrightarrow{L^1_{\text{loc}}} u \text{ as } k \to \infty. $$
Approximations by Explicit Solutions

Pictorial Example

\[ u_l \]

\[ r \]

\[ j = 1 \]

\[ j = 2 \]

\[ j = 3 \]

\[ j = 4 \]
Proposition (HKS 2016)

Let $-\infty < x_1 < \cdots < x_m < \infty$, $l_0 = (-\infty, x_1)$, $\forall i \in \{1, 2, \cdots, m-1\}$ \(l_i = [x_i, x_{i+1})\), and $l_m = [x_m, \infty)$. Let $A_0, \ldots, A_m \in \mathbb{R}$ and, $\forall i \in \{0, \cdots, m-1\}$, $A_i \neq A_{i+1}$ and $b_i = \text{sgn}(A_{i+1} - A_i)$. Then the solution with initial datum $u_I = \sum_{i=0}^{m} A_i \chi_{l_i}$ is given by

$$u(x, t) = \sum_{i=1}^{m-1} \left( \frac{b_i - b_{i-1}}{x_i - x_{i-1}} t + A_i \right) \chi_{l_i}(x) + A_0 \chi_0 + A_m \chi_m$$
Properties

Solution Form

\[ u(x, t) = \sum_{i=1}^{m-1} \left( \frac{b_i - b_{i-1}}{x_i - x_{i-1}} t + A_i \right) \chi I_i(x) + A_0 \chi_0 + A_m \chi_m \]
Properties

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- Only finite mass local minimums/maxima move
Solution Form

\[ u(x, t) = \sum_{i=1}^{m-1} \left( \frac{b_i - b_{i-1}}{x_i - x_{i-1}} t + A_i \right) \chi_I_i(x) + A_0 \chi_0 + A_m \chi_m \]

- Only finite mass local minimums/maximums move
- Will stabilize to a monotone function
Theorem (KLMV 2018)

If $u_I(x)$ is a monotone function on $\mathbb{R}$, then the solution given initial datum $u_I(x)$ is $u(x, t) \equiv u_I(x)$ a.e.
A Degenerate Example

Theorem (KLMV 2018)

If $u_I(x)$ is a monotone function on $\mathbb{R}$, then the solution given initial datum $u_I(x)$ is $u(x, t) \equiv u_I(x)$ a.e.

- The solution doesn’t evolve in time.
Proposition (KLMV 2018)

Let \( u_I \in L^1_{loc}(\mathbb{R}^n) \) be radial, \( u \) be the solution emanating from \( u_I \), and \( J \subset \mathbb{R}^n \) be an annulus or ball centered at 0 such that \( u_I(J) = \{ c \} \) for some constant \( c \in \mathbb{R} \). Then there is a continuous function, \( c(t) : \mathbb{R} \to \mathbb{R} \), such that \( u(J, t) = \{ c(t) \} \).
Results on 1D Data

Pictorial Example: Constancy and Infinite Stabilization Time
Results on 1D Data

Pictorial Example: Constancy and Infinite Stabilization Time

\[ u \]

\[ r \]
Results on 1D Data

Local Monotonicity

Theorem (KLMV 2018)

Let $u_I \in L^1_{\text{loc}}(\mathbb{R})$, $I \subset \mathbb{R}$ be an open interval such that $u_I(I)$ has no local or global extrema, $J \subset \subset I$, and $u$ be the solution emanating from $u_I$. Then there is a $T \in \mathbb{R}$ with $(u|_{J \times (0,T)})(x,t) \equiv u_I(x)$. 
Results on 1D Data

Pictorial Example
Proposition (KLMV 2018)

Let $0 \leq u_1 \in L_{loc}^1(\mathbb{R})$ and $u(x, t) : \mathbb{R} \times (0, T) \to \mathbb{R}$ be the entropy solution emanating from $u_1$. Then the extinction time of $u_1$ is finite if and only if $u_1 \in L^1(\mathbb{R})$, in which case the following formula holds:

$$
\int_{\mathbb{R}} u(x, t) \, dx = \int_{\mathbb{R}} u_1(x) \, dx - 2t \quad \text{for all } 0 \leq t \leq T
$$

- Tracks the loss of mass
- Good theoretical tool to analyze higher dimensional data
Theorem (KLMV 2018)

Suppose \( u \in L^1_{loc}(\mathbb{R}) \) be bounded with compact support. Let \( u^+_I = \max(u_I, 0) \) and \( u^-_I = |\min(u_I, 0)| \). Next, let \( R_n = [x_n, x_{n+1}] \). Finally, denote \( M = \frac{1}{2} \max(\int_{R_1} |u_I| \, dx, \int_{R_2} |u_I| \, dx, \ldots, \int_{R_m} |u_I| \, dx) \). Then, the extinction time of \( u \), \( T \), is governed by the inequality

\[
M \leq T \leq \frac{1}{2} \max\left( \int (u^+_I), \int (u^-_I) \right).
\]
Extinction Time
A Pictorial Example

\[ u_0 \]

\[ x \]
Extinction Time

A Pictorial Example
Proposition (HKS 2016, KLMV 2018)

Suppose that $u(x, t) : \mathbb{R}^N \times (0, \infty) \rightarrow \mathbb{R}$ is an entropy solution with initial datum $u_I \in L^1_{\text{loc}}(\mathbb{R}^N)$. Then for any $M \in \mathbb{N}$, $\tilde{u}(x, y, t) := u(x, t)$, $\tilde{u}$ is an entropy solution with initial datum $\tilde{u}_I \in L^1_{\text{loc}}(\mathbb{R}^N \times \mathbb{R}^M)$ and $\tilde{u}_I(x, y) := u_I(x)$. 
Sheets

Visual Example

![3D Visual Example](image-url)
Corollary (KLMV 2018)

Let $u_I \in L^1_{loc}(\mathbb{R}^n)$ be radial, $f(|x|) := u_I(x)$, $u$ be the solution emanating from $u_I$, and $T$ be the extinction time of $u$ which may be infinite. Let $f^+ := \max(0, f)$, $T^+ := \frac{1}{2} \int_{\mathbb{R}} f^+ \, dx$, $f^- := |\min(0, f)|$, and $T^- := \frac{1}{2} \int_{\mathbb{R}} f^- \, dx$. Then:

$$T \leq \max(T^+, T^-)$$
References


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