General State Sum Construction of Two-Dimensional Topological Quantum Field Theories with Defects

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Our Objective

- problem: finding a general state sum construction for 2-D TQFTs with defects on curves
An Oriented Curve on an Oriented Surface

- start with an oriented curve on an oriented surface
- by convention, we draw the orientation of the surface as counterclockwise
- we triangulate the curve and extend this to a triangulation of the surface
- 3 types of edges (intersect curve at vertex, are on curve, and do not touch curve)
State Sum Construction Conventions

- source model: edges intersecting curve at a point are oriented pointing away from the curve
- orient edges on the curve by the orientation of the curve
- orient edges that do not touch curve by enumerating the vertices
- for each triangle, only one or two edges agree with the boundary orientation, and the other(s) disagree
- triangulations need to be flag-like
Flag-like Triangulation

- three types of edges
- three types of triangles:
Not Flag-like
Triangle Sets

- \( T \) is some triangulation.
- We let \( T^k_n \) represent the set of flag-like \( n \)-simplexes that intersect the curve on \( k \) vertices.
- More particularly, \( T^k_2^+ \) also indicates 2 edges agree with the surface orientation, else \( T^k_2^- \) means 1 edge agrees.
(Extended) Pachner Moves

- needed for invariance and can build all the Alexander moves
- 2D: 2-2 move, 1-3 move
- Extended Pachner: we extend a 1D Pachner move (split line); 2-4
- we require moves to preserve flag-likeness.
2-2 and 1-3 Moves
Counting the Equations

- $3! = 6$
- $3! = 6$
- $2 \times 2 = 4$

- $2 \times 4 = 8$
- $3! = 6$
- $2 \times 2 = 4$
We have 34 equations via the previous slide + the extended Pachner move gives us one unique equation = 35 equations.
State Sum Construction Continued

- 3 sets of labels or colors on edges
  - A: labels on edges that do not intersect curve
  - B: labels on edges that intersect curve at one point
  - C: labels on edges that are on the curve
- 6 sets of coefficients
  - $a, \bar{a}, b, \bar{b}, c, \bar{c}$
  - coefficients without bar are used for the triangles where two edges agree with boundary orientation
  - coefficients with a bar are used for the triangles where one edge agrees with the boundary orientation
  - for each labeling, we multiply these coefficients on all the labeled triangles and we add results over all labelings
Scalar Equation Example 1

\[ \sum_e a_{ue}^r a_{ts}^e = \sum_f a_{fs}^r a_{ut}^f \]
Scalar Equation Example 2

\[
\sum_{e} \overline{c}_u \overline{b}_{e}^{st} = \sum_{f} \overline{c}_f \overline{b}_{u}^{ft}
\]
Scalar Equation Example 3

\[ c_{\tau t}^s = \sum_{\rho, \sigma, \tau} c_{\tau \sigma}^{\tau} b_{\tau \rho}^s \bar{b}_{\tau}^{\sigma \rho} \]
Vector Equations

- observation: our labels should be considered as the basis elements for vector spaces
- 3 vector spaces spanned by each type of label on edges
- each scalar equation is equating coefficients on a vector in the tensor product formed by tensoring outputs that each appear once
Vector Equation Example 1

\[
\begin{aligned}
\sum_r \sum_e a^r_{ue} a^e_{ts} r &= \sum_r \sum_f a^r_{fs} a^f_{ut} r \\
\sum x \otimes y &\mapsto \sum z a^z_{xy} z
\end{aligned}
\]

Notice: This equation expresses associativity of an operation given in terms of a basis by

\[
x \otimes y \mapsto \sum a^z_{xy} z
\]
Vector Equation Example 2

\[ u \mapsto \sum_{r,s,t} \sum_{e} c_{u}^{re} b_{e}^{st} r \otimes s \otimes t = \sum_{r,s,t} \sum_{f} c_{f}^{rs} b_{u}^{ft} r \otimes s \otimes t \]
Vector Equation Example 3

\[ r \otimes t \mapsto \sum_s c_{rt}^s s = \sum_s \sum_{\rho,\sigma,\tau} c_{r\sigma}^\tau b_{\tau\rho}^s \bar{b}_{t}^\rho s \]
String Diagrams

- String diagrams can represent linear maps composed and tensored together to create more linear maps.
- Lines in string diagrams = instances of vector spaces
- Maps are dots that connect these strings
- Read composition from *up* to *down*.
- Strings etc. are tensored *left* to *right*. 
Various Pieces
(Co)actions and (Co)multiplication

Including Defect

Vertex on Defect

The Algebra $A$
Manipulating String Diagrams

- Coherence allows us to manipulate string diagrams by deforming positions along crossed and even unwinding crossings.
- By showing or assuming string diagrams are equivalent, we can locate instances of such string diagrams in other string diagrams and substitute them to simplify equations.
A Sample Substitution

Action Axiom

Substituted in
Frobenius-like

\[ t \otimes f \rightarrow \sum_r \sum_s \sum_e \bar{a}_t^{se} a_r^{ef} s \otimes r \]
\[ t \otimes f \rightarrow \sum_r \sum_s \sum_u \bar{a}_{sr}^u a_{tf}^u s \otimes r \]
\[
\sum_{r} \sum_{s} \sum_{e} \tilde{a}_{t}^{se} a_{ef}^{r} s \otimes r = \sum_{r} \sum_{s} \sum_{u} \tilde{a}_{u}^{sr} a_{tf}^{u} s \otimes r
\]

We get equations touching the curve by paying attention to what set each label came from.
Our Resulting Vector Spaces

1. We verified that $A$ is a special symmetric Frobenius algebra (see Lauda and Pfeiffer 2007).

2. $B$ is a right (co)module over $A$.
   - In fact, if coaction and comultiplication are defined correctly, any module/comodule $X$ over $A$ will work as $B$.

3. $C$ must satisfy equations that also concern $B$ and $A$, but does not necessarily have internal structure of its own.
   - Left (co)action of $C$ on $B$ (co)commutes with (co)action of $A$ on $B$
   - The three spaces must follow certain relations, as will be shown.
We will call $B$ a Frobenius Module if it is both a module and a comodule on some Frobenius algebra $A$, and (in a right Frobenius module) satisfies this relation:
Frobenius-like relations on $\mathcal{C}$
Special-like, needed to satisfy $C$

When we pull out the (co)actions from our loop equations on triangles with an edge as the vertex, we find that these values are the same.
The Extended Pachner Move

\[ \tau = \]
State Sum Overview

- State sum: for triangulation $T$, we consider any coloring $\lambda$
- 6 types = 2 orientations * 3 states of intersection with defect, represented by $a, \bar{a}, b, \bar{b}, c, \bar{c}$
- We take the product of $a, \ldots, \bar{c}$ across all labeled triangles
- then sum all the products for each $\lambda$.
- and from there apply normalization, as allowed by changes in vertex counts (on 2-4 extended move and all the 1-3 moves).
Main Theorem

If $a, \bar{a}, b, \bar{b}, c, \bar{c}$ are the families of structure coefficients for a special symmetric Frobenius algebra $A$, a right Frobenius module $B$ over $A$, and a left coaction and action of $C$ on $B$ (co)commuting with the (co)action of $A$, and these spaces satisfy the string equations shown, then

$$\tau^{-|T_0^1|} \zeta^{-|T_0^0|} \sum_{\lambda} \left( \prod_{\sigma \in T_2^0, +} a(\lambda(\sigma)) \prod_{\sigma \in T_2^0, -} \bar{a}(\lambda(\sigma)) \ldots \prod_{\sigma \in T_2^2, -} \bar{c}(\lambda(\sigma)) \right)$$

where $\lambda$ are all suitable colorings of $T$ and $a(\lambda(\sigma)) \ldots \bar{c}(\lambda(\sigma))$ represents coefficients from the families $a \ldots \bar{c}$ on $\sigma$ with coloring $\lambda$, is independent of triangulation.
Matrix Algebra Example

- The main algebra $A$ relates $n \times n$ square matrices.
- $C$ consists of $m \times m$ matrices.
- $B$ is a right module over $A$ and a left module over $C$ of $m \times n$ matrices.
- Basis elements are (basically) elementary matrices, so the functions are very easily generalized.
- The coefficients again, however, are only 0s and 1s.
\[ A_{ij} \rightarrow \sum_{k=1}^{n} A_{ik} \otimes A_{kj} \]

\[ \rightarrow \sum_{k=1}^{n} A_{ij} = nA_{ij} \]
Summary of Outputs

\[ \zeta = n \]

\[ \zeta A_{ij} \]

\[ nA_{ij} \]
In General

- For the structures described, there are a few more equations to verify
- including some on $C$ and $B$
- but it generalizes easily and it is already well understood that $A$ and $C$ are symmetric Frobenius
Citations

Questions?