Legendrian Weaves

Every knot is a braid closure (Alexander '23), albeit in many ways.

So begin by looking in a nbhd of $S^1$. (Context counterpart: $J'(S^1)$)

Weinstein's nbhd from (ct'd): local model $\mathbb{R}^{2n+1}$, near Legendrian $J'(x)$

$J'(S^1) = T^*S^1 \times \mathbb{R} \quad \alpha = dz - y\,dx \quad \lambda^1 = 0$ Legendrian - e.g. $0 \mapsto (0, df(0)), (0, \epsilon)$

$N = 2$ - strand braid

$\leftarrow$ more efficient encoding

transpositions $T_i$ generate $S_N$
So encode Legendrian surfaces in \( J^+(\mathbb{S}^2) \) by planar diagrams encoding crossing loci of (cusp-less) fronts.

![Planar pieces diagram](image)

**Def.** An \( N \)-graph \( G \) is a collection \( \{ G_1, G_2, \ldots, G_m \} \) of embedded cubic graphs on \( S \) s.t. \( G_i \) intersects \( G_j \) only at vertices, whereupon edges interlace.

**Def.** A Legendrian curve \( \Lambda(\mathbb{S}^2) \) is the Legendrian surface, up to \( 3d \) isotopy, described by an \( N \)-graph \( G \) up to planar isotopy. N.B. 2-graph \( \leftrightarrow \) cubic graph.

Say \( \Lambda \subset J^+(\mathbb{S}^2) \). Define cat. \( C(\Lambda) := \text{Sh}^*(\text{Shx}(\mathbb{A})) \), moduli stack. \( \mu(\Lambda) = 0 \).
1d: Reidemeister III encoded as

2d: Movie of $\mathcal{R}m \circ \mathcal{R}m^{-1}$:

Move a plane:

G

\rightarrow

G

\sim

G

N.B. $V, MV, V_\lambda$

\begin{align*}
V_2 = V_1, & \iff \text{GKS: isotopy of } V_1, V_2 \text{ equiv} \\
V_2 = V_1, & \iff \text{branch point of proj.}
\end{align*}
**Ex:** 2-graphs (cube, planar)

\[ g = 2 \]

\[ \frac{\# P_1}{\# F_1} = q + 1 = 3^2 = * \text{ colors for } \mathbb{Z}/3 \text{ are } \]

\[ * P_{\mathbb{Z}/3} = Q (g - 1)(g - 2) \]

\[ M = \frac{P_g (k)}{Q (g - 1)(g - 2)} \]

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**Moduli Space, M**

12:

\[ V_2 = V \]

22:

\[ V_2 = V \]

\[ \text{N.B.} \quad V, V', V'' \]

\[ r = 0 \]

\[ \text{A flag for each region. Condition on } k \text{'s} \]

\[ \text{Transversality } = \text{ open Bertlmann type} \]

\[ /P(1) \]

\[ \text{stack, but usually with} \]

\[ \text{possible monodromy if non-simp. conn.} \]

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**Planar Structures**

2-graphs

\[ [0-1] \]

\[ \text{chromatic algebra embeds into } \tau \]

\[ \text{correspondence} \]

\[ \text{maps to cohomology} \]

\[ \text{of Bertlmann type (open)} \]

\[ \text{Severel calculus [E-W]} \]

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3-graphs ??

Leads ?

No, seems now:

"verhilding algebra. ??"
Topology

one-cycle \[ \bullet \wedge \bullet = +1 \]

\[ \begin{matrix}
\Delta \quad \text{branches } & \quad \text{one-cycle} & \quad \text{one-cycle}
\end{matrix} \]

(some tree)

\[ \begin{matrix}
\Rightarrow \quad \xrightarrow{\text{connect}} & \quad \xrightarrow{\text{sum}} & \quad \xrightarrow{\text{mutation, flip}}
\end{matrix} \]

\[ Y \Rightarrow Y \quad \text{(connect sum)} \]

Fillings & Clusters

Filling \( \Rightarrow \) cluster chart

\[ \lambda \subset (\mathbb{C}^2)^* \prec \mu \subset \mathbb{C} \]

L: \( \text{coor transform} \rightarrow \text{nonisotopic legs} \)

Surface bounds (2,n) knot

Lagrangian projection goes as Lag filling, embedded if no Reeb chords.

\[ \text{(ideal triangulations of n-gon: } \mathbb{C} \text{n)} \]

Mutation = Lag surgery:

\[ \xrightarrow{\text{sym}} \]

N.B. It gets more complicated!
N-triangulations, spectral networks

\[ \text{monodromy} = \text{cluster word} = \text{triple product of pluggs} \]

Thin: Infinitely many non-isotopic fillings.

* Could not do with 2-graphs

Idea: realize mutations on Lagrangian-filling charts & exploit cluster theory.

Dylan Thurston provided this seq.

\[ \Delta : \begin{array}{c}
1 \\
6 \\
3 \\
5
\end{array} \]

\[ \text{Pot } \Delta = \mu_6 \mu_5 \mu_3 \mu_2 \mu_1 \]

\[ \Delta^n : \begin{array}{c}
1 \\
3 \\
4 \\
5
\end{array} \]

Distinct cluster charts.

Things get complicated quickly!

\[ \Delta \downarrow \text{ local mutation rules} \]

Bryer Knot:

\[ (3, 6) \]

\[ \cdots \text{ and concordances} \]