Function \( x(t), t \in \mathbb{R} \) (or vector-valued: \( \vec{x}(t) = (x_1(t), \ldots, x_n(t)) \))

Action functional \( S(x) := \int L(x(t), \dot{x}(t), t) \, dt \quad \dot{x}(t) := \frac{dx(t)}{dt} \)

\[ \frac{\delta S}{\delta x} = 0 \quad \text{Euler-Lagrange equation} \]

Fact: On \{Solutions\} canonical closed 2-form.

Generalization. (Ostrogradsky):

Same holds for \( L(x, \dot{x}, \ddot{x}, \ldots; t) \) depending on higher derivatives.
Explanation:

Discrete version

continuous time $t \in \mathbb{R}$ $\mapsto$ discrete time $i \in \mathbb{Z}$

For all $i \in \mathbb{Z}$ given $C^0$-mfd $X_i$

replace \( \{ t \mapsto x(t) \} \) by \( \prod_{i \in \mathbb{Z}} X_i \)

replace \( L(\xi(t), \xi(t), \ldots ; t) \) by \( L_i(\xi_i, \xi_{i+1}, \ldots, \xi_{i+k}) \)

up to k-th derivative
Analog of action:

\[ S \left( \left( x_i \in X_i \right)_{i \in \mathbb{Z}} \right) := \sum_i L_i \left( x_i, x_{i+1}, \ldots \right) \]

It is a formal expression, not a functional on configurations\footnote{\( \sum \) diverges)}.

But eqn \( \frac{\delta S}{\delta x_i} = 0 \) makes sense.

\( \Rightarrow \) notion of solutions in infinite discrete time \( \mathbb{Z} \)

Claim (construction) \( \exists \) canonical closed 2-form on \{Solutions\}
$X := \prod_{i \in \mathbb{Z}} X_i$  \quad \text{\textit{$\infty$-dim. manifold}}$

\Omega^\cdot(X) = \otimes_{i} \Omega^\cdot(X_i) \quad \Rightarrow \quad d = \sum_{i \in \mathbb{Z}} d_i$

\text{\textit{$\infty$ many (anti)commuting differentials on $\Omega^\cdot(X)$}}

\underline{Notation} \quad \forall f : \mathbb{Z} \to \mathbb{R}$

$d_{\delta f} := \sum_{i} f(i) d_i$

so \quad d_i = d_{\delta i} \quad d = d_1$
Pick \( f \) such that
\[
\begin{align*}
  f(i) &= 0 & \text{for } i \leq 0 \\
  f(i) &= 1 & \text{for } i > 0
\end{align*}
\]

\[
\omega_S := d_1 \partial S
\]

makes sense \((\text{finite sum})\) and is closed.

Restriction \( \omega_f \big|_{\{\text{Solutions}\}} \) does not depend on \( f \):

Two choices \( f, f' \) in \( f - f' \) has \( \{\text{finite support}\} \)

\( d_2 \partial S \big|_{\{\text{Solutions}\}} = 0 \)

\( d_2 \partial f - f' \) has \( \{\text{finite support}\} \)

\( d_2 \partial \omega_S \big|_{\{\text{Solutions}\}} = 0 \)
Example

\[ S = \sum_{i} L_i(x_i, x_{i+1}) \]

"1st time derivatives"

\[ f = f_{i_0} : = \begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
i_{i_0} & i_{i_0} & i_{i_0+1} & i_{i_0+2} & \ldots & \ldots
\end{array} \]

discrete step function

\[ \omega_f = \frac{d_{i_0}}{d_{i_0+1}} S = d_{i_0} d_{i_0+1} L_{i_0} \]

restriction to \{Solutions\}
does not depend on \( i_0 \in \mathbb{Z} \)

\[ f \text{ as above } \to \text{ class } \in H_c^{\downarrow} (\text{Time}, \mathbb{R}) \]
in sense of coarse geometry
\[ \Omega \in \text{Vert (graph)} \mapsto X_i \text{ smooth} \]

On \( X \) "functional"

\[ S = \sum \text{local terms } S_{\text{edge}} \]

\[ \mathbb{S}^2_{\text{cl}} \left( \left\{ \text{Solutions of EL eqn } S=0 \right\} \right) \leftarrow \text{map} \]

\[ H^1_c (\text{graph: } \mathbb{R}) = \mathbb{R}^2 \]

linear pencil of closed 2-form
Noether Theorem(s)

1) If one has a vector field \( \mathbf{\Sigma} \) on \( \bigotimes_{i=0}^{\infty} \text{local terms} \),

\[
\mathbf{X} = \bigotimes X_i \quad \text{of form} \quad \frac{dx_i}{dt} \text{ for } i = 0, 1, 2, \ldots
\]

which formally preserves \( \mathbf{S} \)

we get a function \( H_\mathbf{S} \) on \( \{ \text{Sols} \} \)
preserved by \( \mathbf{S} \)

"Conservation law" \( H_\mathbf{S} \) can be calculated locally

at any given time \( t_0 \in \mathbb{Z} \).
Write \( s = \sum_{i \in \mathbb{Z}} u_i \frac{\partial}{\partial x_i} \rightarrow s_f := \sum_i f(i) u_i \frac{\partial}{\partial x_i} \)

Pick

\( f := \cdots \cdots \).

\[ \text{Lie}_s \mathcal{S} = \{ H_s, f \} \]

Claim, restriction on \( \{ \text{Solutions} \} \) does not depend on choice of \( f \).

2) (Second Noether Theorem)

if \( \mathcal{S} \) depends linearly on arbitrary function(s)

\[ g : \mathbb{Z} \to \mathbb{R} \]

\( \Rightarrow \) Conservation law is zero.
Moreover: vector fields on $\{\text{solutions}\}$ depending locally on $\lambda$ function $\lambda$ \rightarrow \text{Elements of} \ \ker\Omega^1 \text{isolations.}$

\underline{Examples:} \hspace{1cm} M = \mathbb{R} \times \Sigma^2_{\text{closed oriented surface}} \hspace{1cm} H^1_c(M) = \mathbb{R}$

Action: = Chern-Simons on \{$\text{connections on trivialized bundle}$\}

$A \in \Omega^1(M) \otimes \text{Mat}(n \times n, \mathbb{C})$

$S(A) = \int Tr\left(\frac{A \text{d}A}{2} + \frac{A^3}{3}\right)$ gauge invariant
Solutions of Euler-Lagrange eqn = flat connections
in general, apply to

\[ M = \mathbb{Z}^0 \]

\[ \text{noncompact} \quad \mathbb{R} \]

\[ \text{Ker } \omega = \text{infinitesimal generators of gauge transformations} \]
\[ \omega \text{ is the pullback of a non-degenerate closed 2-form on } \]
\[ \{ \text{Solutions} \} / \text{gauge equivalence} \]

= Stack of representations \( \pi_k(\Sigma) \to \text{GL}(n, \mathbb{C}) \)
\[ \pi_k(\Sigma \times \mathbb{R}) / \text{up to conjugation} \]
Gravity: M space-time of A dimension d
Field = metric g (Euclidean or Lorentzian signature)

Action \( S(g) = \int R(g) \sqrt{\text{det}g} \, d^4x \)

is (formally) \( \text{Diff}(M) \)-invariant

If \( M = (-\varepsilon, \varepsilon) \times N^{d-1} \) compact closed \((d-1)\)-dim. manifold

\( \Rightarrow H^2_c(M, \mathbb{R}) = \mathbb{R} \), get a closed 2-form

on \{Solutions\} \( \ker \omega = \text{Foliation of } \{\text{Sol}\} \) given by \( \text{Vect}(M) \)-action
**Question:** can one generalize construction of $\omega$ by replacing $H^1_c(M) \rightarrow H^2_c(M)$ $\omega \in \Omega^2_c(\text{Solution}) \rightarrow \text{closed 3-form}$?

**Chern-Simons**

In 3-dimensions

$$\text{CS}(A) = \int_{N^4} \text{Tr} \hat{F} \wedge \hat{F}$$

form representing $c_2$

$\dim M = 3$

Each 3-tuple $= 2 \times 4$-tuple

$M = \partial N^4$

Extend 1-form $A$ arbitrarily to 1-form

$A \in \Omega^1(N^4) \otimes \text{Mat}(n \times n, \mathbb{C})$

$$\hat{F} = d\hat{A} + \hat{A}^2 \quad \text{curvature form}$$
5 dimensions

\[ M^5 = \mathbb{R} N^6 \]

\[
\int_{N^6} \text{Tr} \bar{F}_A \bar{F}_A \bar{F}_A = \int_{M^5} \text{CS}_{(r)}(A) = \int_{M^5} \text{Tr} A dA dA dA + \ldots
\]

Take now

\[ M^5 = \mathbb{R}^2 \times \frac{L^3}{\text{compact 3-dim nbd}} \]

\[ \Rightarrow H_2^c(M^5; \mathbb{R}) = \mathbb{R} \]

\[ \omega_{(3)} := d_{\mathbb{R}} d_{\mathbb{R}} d_{\mathbb{R}} S \]

in discrete version

\[ f_1, f_2 : \mathbb{Z} \rightarrow \mathbb{R} \]

\[ f_1 = \begin{cases} 
0 & \text{if } 0 \leq t \leq 0 \\
1 & \text{if } 0 < t \leq 1 \\
0 & \text{if } 1 < t \leq 1 \\
0 & \text{if } t > 1
\end{cases} \]

\[ f_2 = \begin{cases} 
0 & \text{if } 0 \leq t \leq 0 \\
1 & \text{if } 0 < t \leq 0 \\
0 & \text{if } 0 < t \leq 0 \\
0 & \text{if } t > 0
\end{cases} \]
closed 3-form \( \omega_{(3)} \) on \{ \text{Solutions of } \delta S = 0 \}

It now depends on the choice of representative of class \( \in H^2_c(M, \mathbb{R}) \)

---

Claim:

restriction of \( \omega_{(3)} \) to \{ \text{Solutions of } \delta S = 0, \delta^2 S = 0 \}

is canonical!

(does not depend on \( t_1 = 0 \mid \parallel 1 \parallel t_2 = \frac{\parallel 2 \parallel}{0} \)

overdetermined system)
\[ S_S = \int T_r \delta A \wedge F^2 \]

Equations of motion

\[ F^2 = 0 \quad \text{as Mat}(n \times n) \text{-valued 1-form on } M \]

\[ S^2 S = \int T_r \delta A \wedge \delta F \wedge F \]

\[ \delta F = d \delta A + 2 \delta A \cdot A \]

\[ S^2 S = 0 \iff F = 0 \]

flat connections are doubly-critical points
For $M = \mathbb{R}^2 \times L^3$; on $\{\text{flat connections on the trivialized bundles}\}_{\text{on } M}$

we get a closed 3-form.

In fact, it is the pullback of a 3-form on

$\{\text{flat connections}\}_{/\text{gauge equivalence}}$

$= \text{Repr} \left( \pi_1(M) = \pi_1(L^3), \text{GL}(n, \mathbb{C}) \right)$

$\forall$ local system $\mathcal{E}$ on $L^3$: $\mathcal{E}_{\mathcal{E}} = \text{Ext}^1(\mathcal{E}, \mathcal{E})$

$\Lambda^3 \text{Ext}^1(\mathcal{E}, \mathcal{E}) \rightarrow \text{Ext}^3(\mathcal{E}, \mathcal{E})$ $\xrightarrow{\text{Triv}} \mathcal{H}^3(M, \mathbb{C}) = \mathbb{C}$
General situation: action $S$ (sum of local terms)

$\delta S = 0, \delta^2 S = 0, \ldots, \delta^{(k)} S = 0$ \{ overdetemined system \}

$H^k_c \text{ (space-time)} \rightarrow \bigoplus_{i=1}^{k+1} \{ \text{higher critical points} \}$

Chem. Simons in dim $2k+1$

Take $M = \mathbb{R}^k \times L^{k+1}_{\text{closed}} \quad \rightarrow \quad H^k_c(M) = \mathbb{R}$

\$\sim \text{ closed (k+1)-form on } \{ \text{Local systems on } L^{k+1} \}$
Generalization: Topological CS on oriented $(2k+1)$-dim mlds

Holomorphic CS on $(2n+1)$-dim CY varieties

Higher crit point = holomorphic bundles

Return to 3-dim CS, now on compact 3-mlds:

Critical values of $CS_{(3)} \in \mathbb{C}/(2\pi i)^2 \mathbb{Z}$ in ambiguity for nontrivialized bundles.

(all $M, \varnothing \neq 0$): get a countable subgroup.

$K^\text{ind}_3(\mathbb{C}) \cong K^\text{ind}_3(\overline{\mathbb{Q}})$

rigidity conjecture after $\mathbb{R}$

$\overline{\mathbb{Q}}/\overline{\mathbb{Q}} \otimes \mathbb{R}$

Borel Theorem
Another description of the image of regulator map:

A algebraic torus $(\mathbb{C}^*)^m$ with coordinates $z_1, \ldots, z_m$

monomials $z^{\mathbf{r}} := \prod z_i^{r_i} \quad \mathbf{r} \in \mathbb{Z}^m$

\[
\Phi = \sum \left. \text{Li}_2(z^{\mathbf{r}}) \right|_{\text{finite}} + \sum a_{ij} \log z_i \log z_j
\]

subject to $c_0 \in \mathbb{Z}$, $a_{ij} \in \mathbb{Z}$

multivalued function

\{Critical values of $\Phi$\} $\subset$ image of regulator map

Taking all $\Phi$ --- whole image
5-dim CS theory: values of $CS(\mathbb{R}^5)$ on flat connections $/\pi_1^3 \mathbb{Z}$ are special sums of $\text{Li}_3$ (algebraic numbers).

D. Zagier: conjectural description (proved by A. Goncharov).

Claim: Consider $\Phi = \sum c_3 \text{Li}_3(x^3) + \sum \text{Li}_6 \cdot \log + \sum \log \log \log \log \log$

Zagier’s constraint: $\Leftrightarrow \Phi' = 0, \Phi'' = 0$

$\Phi' = 0: \sum \text{Li}_6 \cdot \log = 0$

$\Phi'' = 0: \sum \log \cdots \log \cdots \log$ (symmetric $\text{Sym}^2$)

(not subtracted)
\[ S = CS_{(5)} \]

1. Critical points
   \[ \delta S = 0 \quad \iff \quad F^2 = 0 \]
   \[
   \phi = \sum \lambda_i \eta \quad \text{compare:} \quad \phi' = 0 \quad \text{transcendental equations} \quad \sum \lambda_i \eta = 0
   \]
   Conjecture: same critical values

2. Critical points
   \[ \delta S = 0 \quad \delta^2 S = 0 \quad \iff \quad F = 0 \quad \phi' = 0 \quad \phi'' = 0 \]
   corresponds to each other classes in \( K^\text{ind}_5(\overline{Q}) \)

In 7 dimensions in both situations
   \[ \{1\text{-crit points}\} \supset \{2\text{-crit points}\} \supset \{3\text{-crit points}\} \]
   Conjecture: \( \{1\text{-crit}\} = \{2\text{-crit points}\} \).
On $M^5$: equation $F^2 = 0$ is interesting already for $U(1)$ or $g_2(1)$.

$A \in \Omega^1(M^5): \quad (dA)^2 = 0$

**Question:** How to do perturbation theory at $A = 0$ for $\int e^{\text{const}} \text{Ad}A \text{Ad}A \text{DA} \quad A: U(1)\text{-connection}$?

New physics? (No quadratic part of the action)

~ no Feynman diagrams ...