Enhanced moduli of $D$-branes and superpotentials

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Outline

- based on joint works in progress with D. Calaque, L. Katzarkov, B. Toën, G. Vezzosi, M. Vaquié
- shifted symplectic geometry
- foliations in derived geometry
- isotropic and Lagrangian foliations
- quotients by foliations and derived Darboux theorems
- examples and constructions
Symplectic structures

**Recall:** For $X$ a smooth space/$\mathbb{C}$, a *symplectic structure* is an $\omega \in H^0(X, \Omega^2_X)$ such that its adjoint $\omega^b : T_X \to \Omega^1_X$ is a sheaf isomorphism.

**Note:** Does not work for $X$ singular (or stacky or derived):

- $T_X$ and $\Omega^1_X$ are too crude as invariants and get promoted to complexes $T_X$ and $\mathbb{L}_X$.
- A form being closed is not just a condition but rather an extra structure.
**Definition:** $X$ derived Artin stack locally of finite presentation (so that $\mathbb{L}_X$ is perfect).

- A $n$-shifted 2-form $\omega : \mathcal{O}_X \to \mathbb{L}_X \wedge \mathbb{L}_X[n]$ - i.e. $\omega \in \pi_0(\mathcal{A}^2(X; n))$ - is nondegenerate if its adjoint $\omega^b : \mathbb{T}_X \to \mathbb{L}_X[n]$ is an isomorphism (in $D_{qcoh}(X)$).

- The space of $n$-shifted symplectic forms $\text{Sympl}(X; n)$ on $X/\mathbb{C}$ is the subspace of $\mathcal{A}^{2,\text{cl}}(X; n)$ of closed 2-forms whose underlying 2-forms are nondegenerate i.e. we have a homotopy cartesian diagram of spaces

\[
\begin{array}{ccc}
\text{Sympl}(X, n) & \longrightarrow & \mathcal{A}^{2,\text{cl}}(X, n) \\
\downarrow & & \downarrow \\
\mathcal{A}^{2}(X, n)^{nd} & \longrightarrow & \mathcal{A}^{2}(X, n)
\end{array}
\]
Shifted symplectic structures: examples (i)

- Nondegeneracy: a duality between the stacky (positive degrees) and the derived (negative degrees) parts of $\mathbb{L}_X$.

- $G = GL_n \rightarrow BG$ has a canonical 2-shifted symplectic form whose underlying 2-shifted 2-form is

$$k \rightarrow (\mathbb{L}_{BG} \wedge \mathbb{L}_{BG})[2] \simeq (\mathfrak{g}^\vee [-1] \wedge \mathfrak{g}^\vee [-1])[2] = \text{Sym}^2 \mathfrak{g}^\vee$$

given by the dual of the trace map $(A, B) \mapsto tr(AB)$.

- Same as above (with a choice of $G$-invariant symm bil form on $\mathfrak{g}$) for $G$ reductive over $k$.

- The $n$-shifted cotangent bundle $T^\vee X[n] := \text{Spec}_X(\text{Sym}(\mathbb{T}_X[-n]))$ has a canonical $n$-shifted symplectic form.
Shifted symplectic structures: examples (ii)

**Theorem:** [PTVV] Suppose:

- \((F, \omega)\) is an \(n\)-shifted symplectic stack.
- \(X\) is an \(\mathcal{O}\)-compact derived stack equipped with an \(\mathcal{O}\)-orientation \([X] : \mathbb{H}(X, \mathcal{O}_X) \to \mathbb{C}[-d]\) of degree \(d\).

If Map\((X, F)\) is a derived Artin stack locally of finite presentation over \(\mathbb{C}\), then Map\((X, F)\) carries a canonical \((n - d)\)-shifted symplectic structure.

**Remark:**

0) Analogue of the Alexandrov-Kontsevich-Schwarz-Zaboronsky result.

1) \(\mathcal{O}\)-orientation on \(X\) of degree \(d = \text{Calabi-Yau structure of dimension} \ d\);

2) A **compact oriented topological** \(d\)-manifold has an \(\mathcal{O}\)-orientation of degree \(d\) (Poincaré duality).
Lagrangian structures

Let \((Y, \omega)\) be a \(n\)-shifted symplectic derived stack. A lagrangian structure on a map \(f : X \to Y\) is

- path \(\gamma\) in \(\mathcal{A}^{2,\text{cl}}(X; n)\) from \(f^*\omega\) to 0
- that is 'non-degenerate' (in a suitable sense), i.e. the induced map \(\theta_\gamma : \mathbb{T}_f \to \mathbb{L}_X[n - 1]\) is an equivalence.

Examples:

- usual smooth lagrangians \(L \hookrightarrow (Y, \omega)\) where \((Y, \omega)\) is a smooth \((0)\)-symplectic scheme.
- there is a bijection between lagrangian structures on the canonical map \(X \to (\text{Spec } k, \omega_{n+1})\) and \(n\)-shifted symplectic structures on \(X\) (thus lagrangian structures generalize shifted symplectic structures)
**Theorem:** [PTVV] Let \((F, \omega)\) be \(n\)-shifted symplectic derived Artin stack, and \(L_i \to F\) a map of derived stacks equipped with a Lagrangian structure, \(i = 1, 2\). Then the homotopy fiber product \(L_1 \times_F L_2\) is canonically a \((n - 1)\)-shifted derived Artin stack.

In particular, if \(F = Y\) is a smooth symplectic Deligne-Mumford stack (e.g. a smooth symplectic variety), and \(L_i \hookrightarrow Y\) is a smooth closed lagrangian substack, \(i = 1, 2\), then the derived intersection \(L_1 \times_F L_2\) is canonically \((-1)\)-shifted symplectic.
**Remark:** An interesting case is the derived critical locus $\mathbb{R} \text{Crit}(f)$ for $f$ a global function on a smooth symplectic Deligne-Mumford stack $Y$. Here

$$
\begin{array}{ccc}
\mathbb{R} \text{Crit}(f) & \longrightarrow & Y \\
\downarrow & & \downarrow df \\
Y & \longrightarrow & T^\vee Y \\
0 & \longrightarrow & 
\end{array}
$$
Local models (i)

**Recall:** In classical symplectic geometry the local structure of a symplectic manifold is described by the **Darboux theorem:**
Local models (i)

**Recall:** In classical symplectic geometry the local structure of a symplectic manifold is described by the **Darboux theorem:** a *symplectic structure is locally (in the $C^\infty$ or analytic setting) or formally (in the algebraic setting) isomorphic to the standard symplectic structure on a cotangent bundle.*
Local models (i)

In the derived and stacky setting there are two natural incarnations of an $n$-shifted symplectic cotangent bundle:

(a) The shifted cotangent bundle $T^\vee_M[n] = \mathbb{R} \text{Spec}_{/M} (\text{Sym}_{\mathcal{O}_M} (T_M[-n]))$, equipped with $n$-th shift of the standard symplectic form;

(b) The derived critical locus $\text{Rcrit}(w)$ of an $n + 1$ shifted function $w : M \to \mathbb{A}^1[n + 1]$, equipped with the inherited $n$-shifted symplectic form $\omega_{\text{Rcrit}(w)}$.

defined for general Lagrangian intersections in [PTVV'2012].
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Note: (a) is a special case of (b) corresponding to the zero shifted function.
Local models (ii)

Remark:  • Shifted cotangent bundles are too restrictive to serve as local models of shifted symplectic structures.
• Derived critical loci of shifted functions have enough flexibility to provide local models. This leads to a remarkable shifted version of the Darboux theorem:
Local models (ii)

**Theorem:** [BBBJ’2013] Let $X$ be a derived Deligne-Mumford stack, and let $\omega$ be an $n$-shifted symplectic structure on $X$, with $n \leq 0$.

Then, étale locally $(X, \omega)$ is isomorphic to $(R_{\text{crit}}(w), \omega_{R_{\text{crit}}(w)})$ for some shifted function $w : M \to \mathbb{A}^1[n+1]$ on a derived scheme $M$.

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Local models (ii)

**Theorem:** [BBBJ’2013] Let $X$ be a derived Deligne-Mumford stack, and let $\omega$ be an $n$-shifted symplectic structure on $X$, with $n \leq 0$.

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**Question:** Find additional geometric structures that will ensure a global existence of a potential?
Local models (iii)

**Answer:** Potentials always exist in the presence of isotropic foliations.
Local models (iii)

**Theorem:** Let $X$ be a derived stack, locally of f.p. and let $\omega$ be an $n$-shifted symplectic structure on $X$. Assume:
- $\omega$ is exact, i.e. $[\omega] = 0 \in H^*_{DR}(X)$;
- $(X, \omega)$ is equipped with an isotropic foliation $(\mathcal{L}, h) = (L, \alpha, \epsilon; h)$.

Then there exists
- a shifted function $f : [X/\mathcal{L}] \to \mathbb{A}^1[n + 1]$, and
- a symplectic map $s : X \to \text{Rcrit}(f)$ of $n$-shifted symplectic stacks, i.e. $s^*\omega_{\text{Rcrit}(f)} = \omega$.

Moreover, if $(\mathcal{L}, h)$ is Lagrangian, then $s$ is étale.
Local models (iv)

Note: This connects directly to the [BBBJ’2013] Darboux theorem because of the following result:

**Theorem:** [PTVV] Let $X$ be a derived stack, locally of f.p. and let $\omega$ be any $n$-shifted closed $p$-form on $X$ with $n < 0$. Then $\omega$ is exact, i.e. $[\omega] = 0 \in H^\bullet_{DR}(X) = \mathbb{H}^\bullet(\mathcal{A}^{0,\text{cl}}(X))$.

Note: $[\omega] \in H^{p+n}_{DR}(X)$ and in general $H^{p+n}_{DR}(X) \neq 0$. So the statement is not a triviality.
Examples (i)

**Example (1) Derived critical loci.** Let $Z$ be a smooth scheme, $w: Z \to \mathbb{A}^1$ a regular function. Consider $X = \text{Rcrit}(w)$ with its inherited $(-1)$-shifted symplectic structure $\omega_{\text{Rcrit}(w)}$. Let $\iota: X \to Z$ be the natural map, and let $L_{\iota} = (\mathbb{L}_{X/Z}, \text{res}, d_{\text{DR}})$ be the associated tangential foliation. Then:

**Claim:**

- The foliation $L_{\iota}$ has a natural Lagrangian structure $h$.
  - The quotient $[X/L_{\iota}] = \hat{\text{Z}}_{\text{crit}(w)}$ is the formal completion of $Z$ along $\text{crit}(w) = t_0(X)$.
  - The potential $f: \hat{\text{Z}}_{\text{crit}(w)} \to \mathbb{A}^1$ associated with $h$ is given by $f = w|_{\hat{\text{Z}}_{\text{crit}(w)}}$. 
Examples (ii)

Variant: If $Z \in \text{dSt}_\mathbb{C}$ is a derived stack locally of finite type, $w : Z \to \mathbb{A}^1[n]$ is an $n$-shifted function, and $X = \text{Rcrit}(w) \to Z$, then

Claim:  
- The foliation $\mathcal{L}_i$ has a natural Lagrangian structure $h$.
- The quotient $[X/\mathcal{L}_i] = \hat{X}_i$ is the relative completion of $X$ along $i$.
- The potential $f : \hat{Z}_{\text{crit}(w)} \to \mathbb{A}^1[n]$ associated with $h$ is given by $f = w|_{\hat{X}_i}$. 

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Examples (iii)

(2) Cotangent bundles. If $M$ is a smooth manifold, and

- $X = T^\vee M$,
- $\omega = \text{(the standard symplectic structure)}$.

Then: The natural projection $\pi : X \rightarrow M$ gives rise to a tangential foliation $\mathcal{L}_{\pi} = (L_{\pi}, \text{res}, d_{DR})$ which is Lagrangian.

In this case:

- $[X/\mathcal{L}_{\pi}] = (X/M)_{DR}$,
- $f = 0$ viewed as a 1-shifted function,

and we get an identification $\text{Rcrit}(f) = T^\vee_M [1 - 1] = X$ together with the natural 0-shifted symplectic forms.
Examples (iv)

(3) Twisted cotangent bundles. Suppose $M$ is a smooth manifold over $\mathbb{C}$ and
\[ \eta \in H^1(M, \Omega^\geq_1 M[1]) = H^2 \left( M, \Omega^1_M \xrightarrow{d} \Omega^{2,\text{cl}}_M \right). \]
Such $\eta$ gives rise to an algebraic symplectic manifold - the twisted cotangent bundle $(\pi_\eta : X_\eta \to M, \omega_\eta)$.

Note:
- The tangential foliation $\mathcal{L}_{\pi_\eta}$ is Lagrangian.
- If $\omega_\eta$ is exact, then $(X_\eta, \omega_\eta)$ will be symplectically isomorphic to $R_{\text{crit}} (f)$ for a 1-shifted function $f$ on $[X_\eta/\mathcal{L}_{\pi_\eta}] = (X_\eta/M)_{\text{DR}}$. 
Examples (v)

We are looking for a shifted function \( f : (X_\eta/M)^{DR} \rightarrow \mathbb{A}^1[1] \), or equivalently for an element

\[
f \in H^1(M, \mathcal{H}^{\bullet}_{DR}(X/M)) = H^1(M, \mathcal{O}_M).
\]

By construction \( [\omega_\eta] = 0 \in H^2_{DR}(X_\eta) \) if and only if \( \eta \) is in the image of the map \( d : H^1(M, \mathcal{O}_M) \rightarrow H^1(M, \mathcal{O}_M^{\geq 1}[1]) \). Therefore \( \omega_\eta \) is exact precisely when we can find \( f \in H^1(M, \mathcal{O}_M) \) such that \( \eta = df \). This \( f \) is the shifted function provided by the theorem, i.e.

\[
(X_\eta, \omega_\eta) \cong (R\text{crit}(f), \omega_{R\text{crit}(f)}).
\]

**Note:** Note that as in the classical case \( f \) is only unique up to a class in \( H^1(M, \mathbb{C}) \), i.e. up to a (locally) constant 1-shifted function on \( (X_\eta/M)^{DR} \).
Examples (vi)

(4) Integrable systems. Let \((X, \omega)\) be an exact symplectic manifold, and let

\[ h : X \to B \]

be a smooth completely integrable system structure on \(X\). Again the tangential foliation \(\mathcal{L}_h\) is Lagrangian and \([X/\mathcal{L}_h] = (X/B)_{DR}\) and by the theorem we can find

\[ f : (X/B)_{DR} \to \mathbb{A}^1[1] \]

such that \((X, \omega) = (\text{Rcrit}(f), \omega_{\text{Rcrit}(f)})\).
Examples (vii)

Now note that

\[
\text{Map}_{\text{dSt}_C} \left((X/B)_{\text{DR}}, \mathbb{A}^1[1]\right) \rightarrow H^1 \left(B, \mathcal{H}^\bullet_{\text{DR}}(X/B)\right)
\]

\[
\cup H^0 \left(B, h_* \Omega^1_{X/B}\right)
\]

If \( \lambda \in H^0 \left(X, \Omega^1_X\right) \) is such that \( \omega = d\lambda \), then \( \lambda \) maps to a relative 1-form \( \lambda^{rel} \in H^0 \left(X, \Omega^1_{X/B}\right) = H^0 \left(B, h_* \Omega^1_{X/B}\right) \). One now checks that \( f = \lambda^{rel} \).

Note: The full form \( \lambda \) also plays a role in the picture. It defines the map \( s : X \rightarrow \text{Rcrit}(f) \).
Examples (viii)

Indeed, if $A$ is a reduced $\mathbb{C}$-algebra, then $(X/B)_{DR}(A) = X(A)$, i.e. $((X/B)_{DR})_{\text{red}} = X$. In particular $f|_{((X/B)_{DR})_{\text{red}}} = 0$ as it is the image of $f = \lambda^{\text{rel}} \in H^0 \left( B, h_* \Omega^1_{X/B} \right) \subset H^1 \left( B, \mathcal{H}^\bullet_{DR}(X/B) \right)$ in $H^1(X, \mathcal{O})$.

Therefore $\text{Rcrit}(f)(A) = \text{Rcrit}(0)(A) = T^\vee X(A)$ i.e. $\text{Rcrit}(f)_{\text{red}} = T^\vee X$.

Since $X$ itself is reduced, the map $s : X \to \text{Rcrit}(f)$ will factor as

$\begin{tikzcd}
X & \text{Rcrit}(f)_{\text{red}} \ar[l, bend left] & \text{Rcrit}(f) \\
& s \ar[u, hookrightarrow] &
\end{tikzcd}$

and it can be checked that the map $X \to T^\vee X$ coincides with the section $\lambda$. 
Higher Chern-Simons functionals (i)

Let $M$ be a compact oriented $C^\infty$ manifold of dimension $d = 2k + 1$. Choose a Morse-Smale function $\mu : M \to \mathbb{R}$.

A self-indexing Morse function, i.e. for every $x \in \text{crit}(\mu)$ we have $\mu(x) = \text{ind}_\mu(x)$. 

**Higher Chern-Simons functionals (i)**

Let $M$ be a compact oriented $C^\infty$ manifold of dimension $d = 2k + 1$. Choose a Morse-Smale function $\mu : M \to \mathbb{R}$. Choose $c \in (k, k + 1)$, and let $M^+ := \mu^{-1}((\infty, c])$. Then

- $M^+$ is a manifold with boundary;
- the inclusion $M^+ \hookrightarrow M$ induces a homotopy equivalence between $M^+$ and the $k$-dimensional skeleton of $M$.

Fix a complex reductive group $G$, and let $\text{Bun}_G(M) = \text{Map}_{d\text{St}_G}(M, BG)$ be the derived moduli stack of $G$-local systems on $M$. By [PTVV] $\text{Bun}_G(M)$ carries a natural $2 - d$-shifted symplectic structure $\omega$, and so if $k \geq 1$, it follows that $\omega$ is exact.
Higher Chern-Simons functionals (ii)

**Theorem:** [KPTVV] The tangential foliation for the restriction morphism

\[ \text{res}^+ : \text{Bun}_G(M) \to \text{Bun}_G(M^+) \]

can be equipped with a natural isotropic structure \( h \) which depends only on the orientation data of \( M \) and the shifted symplectic form on \( BG \).

Hence we can find a shifted function

\[ f : \left( \text{Bun}_G(M) / \text{Bun}_G(M^+) \right)_{DR} \to \mathbb{A}^1[2 - 2k] \]

and a symplectic map

\[ s : (\text{Bun}_G(M), \omega) \to (\text{Rcrit}(f), \omega_{\text{Rcrit}(f)}) . \]
Potentials in non-abelian Hodge theory (i)

Let $M$ be a smooth projective variety with $\dim_{\mathbb{C}} M = d$ and consider the derived stack of rank $n$ local systems on $M$:

$$X := \text{Loc}_n(M) = \text{Map}_{d\text{St}_{\mathbb{C}}}(M, B\text{GL}_n).$$

From [PTVV’2012] we know that $X$ is equipped with a natural $(2 - 2d)$-shifted symplectic structure $\omega_X$. This symplectic structure comes with natural refinements:

- $T_X$ has a natural Hodge filtration.
- $(X, \omega_X)$ is the general fiber of a $\mathbb{C}^\times$ twisted symplectic family $(\mathcal{X}, \omega_{\mathcal{X}/\mathbb{A}^1}) \to \mathbb{A}^1$ of moduli of $\lambda$-connections, and on tangent complexes this gives the standard Hodge filtration.
This implies

**Claim:** The natural map \( \omega^b_X : \mathbb{T}_X \to \mathbb{L}_X[2 - 2d] \) given by \( \omega_X \) is a filtered quasi-isomorphism for the Hodge filtrations.

As a consequence in the middle degree one gets:

**Theorem:** When \( d = 2k + 1 \), the natural map

\[
F^{k+1}\mathbb{T}_X \to \mathbb{T}_X
\]

admits a canonical structure of a Lagrangian foliation. In particular \((X, \omega_X)\) (and the corresponding Higgs moduli) are identified with the critical locus of a shifted potential.
Potentials in non-abelian Hodge theory (iii)

Remarks:

(1) The foliation $F^{k+1}T_X \to T_X$ is the tangential foliation for the map $\text{res}^{\leq k+1} : X \to \text{Loc}_{n}^{\leq k}(M)$.

derived moduli stack of dg modules over $\left(\Omega_{\mathcal{M}}^{\leq k}, d\right)$ which are locally free of rank $n$
Potentials in non-abelian Hodge theory (iii)

Remarks:

(1) The foliation $F^{k+1}T_X \to T_X$ is the tangential foliation for the map $\text{res}^{\leq k+1} : X \to \text{Loc}_n^{\leq k}(M)$.

(2) If $k = 0$, then $\text{Loc}_n^{\leq 0}(M) = \text{Bun}_n(M)$ and the map $\text{res}^{\leq 0} : X \to \text{Bun}_n(M)$ is a twisted cotangent bundle.

(3) If $k \geq 1$, then the map induces an isomorphism of truncations $t_{\geq -k}\text{Loc}_n(M) \to t_{\geq -k}\text{Loc}_n^{\leq k}(M)$.

(4) The full untrancated stack $X = \text{Loc}_n(M)$ is recovered as a critical locus of a shifted function on $(\text{Loc}_n(M)/\text{Loc}_n^{\leq k}(M))_{DR}$ which can be checked again comes from an element $f \in H^{2-2k}(\text{Loc}_n^{\leq k}(M), \mathcal{O})$. 
**Tangent complex**

\[ X \in \text{dSt}_\mathbb{C}, \ x : \text{Spec}(\mathbb{C}) \to X \text{ a point} \]

\[
\left( \text{Stalk } \mathbb{T}_{X,x} \text{ of the tangent complex} \right) = \left( \text{normalized chain complex of the homotopy fiber of } X(\mathbb{C}[\varepsilon]) \to X(\mathbb{C}) \text{ over } x \right)
\]

simplicial abelian group
Tangent complex

\[ X \in dSt_C, \ x : \text{Spec}(\mathbb{C}) \rightarrow X \text{ a point} \]

\[
\left( \text{Stalk } \mathbb{T}_{X,x} \text{ of the tangent complex} \right) = \left( \text{normalized chain complex of the homotopy fiber of } \right)
\]

\[ X(\mathbb{C}[\varepsilon]) \rightarrow X(\mathbb{C}) \text{ over } x \]

When \( X \) is a moduli stack:

\[ H^{-1}(\mathbb{T}_{X,x}) = \text{infinitesimal automorphisms of } x; \]

\[ H^0(\mathbb{T}_{X,x}) = \text{infinitesimal deformations of } x; \]

\[ H^1(\mathbb{T}_{X,x}) \supseteq \text{obstructions of } x. \]
### Examples

- $X = BG = [pt / G] \Rightarrow T_{X, pt} = \mathfrak{g}[1]$.

- $X = \text{derived intersection } L_1 \times^h_M L_2 = (L_1 \cap L_2, \mathcal{O}_{L_1} \otimes_{\mathcal{O}_M} \mathcal{O}_{L_2})$ of smooth subvarieties $L_1, L_2 \subset M$ in a smooth $M \Rightarrow T_{X, x} = [T_{L_1, x} \oplus T_{L_2, x} \to T_{M, x}]$, $H^0(T_{X, x}) = T_{L_1 \cap L_2, x}$; $H^1(T_{X, x}) = \text{failure of transversality}$.

- $X = \text{moduli of vector bundles } E \text{ on a smooth projective } Y \Rightarrow T_{X, E} = R\Gamma(Y, \text{End}(E))[1]$.

- $X = \text{moduli of maps } f \text{ from } C \text{ to } Y \Rightarrow T_{X, f} = R\Gamma(C, f^* T_Y)$.

- $X = \text{moduli of local systems } E \text{ on a compact manifold } Y \Rightarrow T_{X, E} = R\Gamma(Y, \text{End}(E))[1]$. 
**Cotangent complex**

\[ A \in \text{cdga}_{\mathbb{C}}, \quad X = \mathbb{R}\text{Spec}(A) \in \text{dSt}_{\mathbb{C}}, \]
\[ A' \to A \text{ a cofibrant (semifree) replacement} \]

\[
\begin{pmatrix}
\text{cotangent complex} \\
L_X = L_A
\end{pmatrix}
= 
\begin{pmatrix}
\text{Kähler differentials} \\
\Omega^1_{A'} \text{ of } A'
\end{pmatrix}
\]

If \( X \in \text{dSt}_{\mathbb{C}} \) is a general derived Artin stack, then
\[ X = \text{hocolim}\{\mathbb{R}\text{Spec } A \to X\} \] (in the model category \( \text{dSt}_{\mathbb{C}} \)) and
\[ L_X = \text{holim}_{\mathbb{R}\text{Spec } A \to X} L_A \]

**Note:**

- \( L_X \in L_{qcoh}(X) \) - the dg category of quasi-coherent \( \mathcal{O}_X \) modules.
- \( X \) is locally of finite presentation iff \( L_X \) is perfect. In this case
\[ T_X = L_X^\vee = \text{Hom}(L_X, \mathcal{O}_X). \]
$p$-forms

$A \in \text{cdga}_\mathbb{C}, \quad X = \mathbb{R}\text{Spec}(A) \in \text{dSt}_\mathbb{C},$

$A' \to A$ a cofibrant (semifree) replacement. Then:

$$\bigoplus_{p\geq 0} \bigwedge_A^p \mathbb{L}_A = \bigoplus_{p\geq 0} \Omega^p_{A'} - \text{a fourth quadrant bicomplex with}$$

vertical differential $d : \Omega^p_{A'} \rightarrow \Omega^{p,i+1}_{A'}$ induced by $d_{A'}$, and

horizontal differential $d_{DR} : \Omega^p_{A'} \rightarrow \Omega^{p+1,i}_{A'}$ given by the de Rham differential.

**Hodge filtration:** $F^q(A) := \bigoplus_{p>q} \Omega^p_{A'}$: still a fourth quadrant bicomplex.
(shifted) closed $p$-forms

**Motivation:** If $X$ is a smooth scheme/$\mathbb{C}$, then $\Omega^p_{X, \text{cl}} \cong (\Omega^\geq p_X, d)$. Use $(\Omega^\geq p_X, d)$ as a model for closed $p$ forms in general.

**Definition:**

- complex of closed $p$-forms on $X = \mathbb{R}\text{Spec } A$:
  \[ A^{p,cl}(A) := \text{tot} \prod (F^p(A))[p] \]

- complex of $n$-shifted closed $p$-forms on $X = \mathbb{R}\text{Spec } A$:
  \[ A^{p,cl}(A; n) := \text{tot} \prod (F^p(A))[n + p] \]

- Hodge tower:
  \[ \cdots \to A^{p,cl}(A)[-p] \to A^{p-1,cl}(A)[1-p] \to \cdots \to A^{0,cl}(A) \]
(shifted) closed $p$-forms (ii)

Explicitly an $n$-shifted closed $p$-form $\omega$ on $X = \mathbb{R} \text{Spec } A$ is an infinite collection

$$\omega = \{\omega_i\}_{i \geq 0}, \quad \omega_i \in \Omega_A^{p+i,n-i}$$

satisfying

$$d_{DR} \omega_i = -d \omega_{i+1}.$$

**Note:** The collection $\{\omega_i\}_{i \geq 1}$ is the key closing $\omega$. 
**p-forms and closed p-forms**

**Note:**
- The complex $A^{0,cl}(A)$ of closed 0-forms on $X = \mathbb{R}Spec A$ is exactly Illusie’s derived de Rham complex of $A$.
- There is an underlying $p$-form map

$$A^{p,cl}(A; n) \to \wedge^p \mathbb{L}_{A/k}[n]$$

inducing

$$H^0(A^{p,cl}(A)[n]) \to H^n(X, \wedge^p \mathbb{L}_{A/k}).$$

- The homotopy fiber of the underlying $p$-form map can be very complicated (complex of keys): being closed is *not* a property but rather a list of coherent data.
Functoriality and gluing:

- the \( n \)-shifted \( p \)-forms \( \infty \)-functor
  \[ \mathcal{A}^p(-; n) : \text{cdga}_C \to \text{SSets} : A \mapsto | \Omega^p_{QA}[n] \cong (\wedge^p_A)_A[n] | \]

- the \( n \)-shifted closed \( p \)-forms \( \infty \)-functor
  \[ \mathcal{A}^{p,\cl}(-; n) : \text{cdga}_C \to \text{SSets} : A \mapsto | A^{p,\cl}(A)[n] | \]

- \( \mathcal{A}^p(-; n) \) and \( \mathcal{A}^{p,\cl}(-; n) \) are derived stacks for the \( \acute{e}tale \) topology.

- underlying \( p \)-form map (of derived stacks)
  \[ \mathcal{A}^{p,\cl}(-; n) \to \mathcal{A}^p(-; n) \]

**Notation:** \( |-| \) denotes \( \text{Map}_{\mathbb{C}-\text{dgMod}}(\mathbb{C}, -) \) i.e. Dold-Kan applied to the \( \leq 0 \)-truncation - all our dg-modules have cohomological differential
global forms and closed forms

For a derived Artin stack $X$ (locally of finite presentation $/_\mathbb{C}$) we have

**Definition:**

- $A^p(X) := \text{Map}_{dSt_{/\mathbb{C}}}(X, A^p(-))$ is the space of $p$-forms on $X$;
- $A^{p,\text{cl}}(X) := \text{Map}_{dSt_{/\mathbb{C}}}(X, A^{p,\text{cl}}(-))$ is the space of closed $p$-forms on $X$;
- the corresponding $n$-shifted versions:
  - $A^p(X; n) := \text{Map}_{dSt_{/\mathbb{C}}}(X, A^p(-; n))$
  - $A^{p,\text{cl}}(X; n) := \text{Map}_{dSt_{/\mathbb{C}}}(X, A^{p,\text{cl}}(-; n))$
- an $n$-shifted (resp. closed) $p$-form on $X$ is an element in $\pi_0 A^p(X; n)$ (resp. in $\pi_0 A^{p,\text{cl}}(X; n)$)
global forms and closed forms (ii)

**Note:** If $X$ is a smooth scheme there are no negatively shifted forms; when $X = \mathbb{R}Spec A$ then there are no positively shifted forms. For general $X$ they might exist for any $n \in \mathbb{Z}$. 
global forms and closed forms (ii)

- **underlying $p$-form map**: $\mathcal{A}^{p,\text{cl}}(X; n) \to \mathcal{A}^p(X; n)$
- not a monomorphism for general $X$: its homotopy fiber at a given $p$-form $\omega_0$ is the space of *keys* of $\omega_0$.
- If $X$ is a smooth and proper scheme then this map is a monomorphism (homotopy fiber is either empty or contractible) $\Rightarrow$ no new phenomena in this case.
- **Theorem (PTVV)**: for $X$ derived Artin, $\mathcal{A}^p(X; n) \simeq \text{Map}_{\text{Lqcoh}}(X)(\mathcal{O}_X, (\wedge^p \mathbb{L}_X)[n])$ (smooth descent)
- in particular a $n$-shifted $p$-form on $X$ is an element in $H^n(X, \wedge^p \mathbb{L}_X)$
Examples (i):

(1) If $X = \text{Spec}(A)$ is an usual (underived) smooth affine scheme, then

$$A^{p,\text{cl}}(X; n) = (\tau_{\leq n}(\Omega_A^p \xrightarrow{d_{\text{DR}}} \Omega_A^{p+1} \xrightarrow{d_{\text{DR}}} \cdots ))[n],$$

and hence

$$\pi_0 A^{p,\text{cl}}(X; n) = \begin{cases} 
0, & n < 0 \\
\Omega_A^{p,\text{cl}}, & n = 0 \\
H^{n+p}_{\text{DR}}(X), & n > 0
\end{cases}$$

e.g. if $X = \mathbb{C}^\times$, then $dz/z \in \pi_0 A^{1,\text{cl}}(X; 0)$ and also $dz/z \in \pi_0 A^{0,\text{cl}}(X; 1)$.
Examples (ii):

(2) If $X$ is a smooth and proper scheme, then
$$\pi_i \mathcal{A}^{p,cl}(X; n) = F_p H^{n+p-i}_{DR}(X).$$

(3) If $X$ is a (underived) higher Artin stack, and $X_\bullet \to X$ is a smooth affine simplicial groupoid presenting $X$, then
$$\pi_0 \mathcal{A}^p(X; n) = H^n(\Omega^p(X_\bullet), \delta)$$
with $\delta = \check{\text{Čech}}$ differential.
In particular if $G$ is a complex reductive group, then
$$\pi_0 \mathcal{A}^p(BG; n) = \begin{cases} 0, & n \neq p \\ (\text{Sym}^\bullet g^\vee)^G, & n = p. \end{cases}$$
Examples (iii):

(4) Similarly

\[ \mathcal{A}^{p,cl}(BG; n) = \left| \prod_{i \geq 0} (\text{Sym}^{p+i} \mathfrak{g}^\vee)^G [n + p - 2i] \right|, \]

and so

\[ \pi_0 \mathcal{A}^{p,cl}(BG; n) = \begin{cases} 
0, & \text{if } n \text{ is odd} \\
(\text{Sym}^{p} \mathfrak{g}^\vee)^G, & \text{if } n \text{ is even.}
\end{cases} \]
Examples (iv): (5) If $X = \text{Rzero}(s)$ for $s \in H^0(L, E)$ on a smooth $L$, then

$$\Omega^1_X = E|_Z \xrightarrow{(\nabla s)^b} \Omega^1_{L|Z},$$

and if we choose $\nabla$ local flat algebraic connection on $E$ we can rewrite $\Omega^1_X$ as a module over the Koszul complex:
Examples (v):

In the same way we can describe $\Omega_X^2$ as a module over the Koszul complex

\[
\cdots \longrightarrow \wedge^2 E^\vee \otimes \Omega_L^2 \longrightarrow E^\vee \otimes \Omega_L^2 \longrightarrow \Omega_L^2 \longrightarrow \Omega_L^2|_Z \longrightarrow 0
\]

\[
\cdots \longrightarrow \wedge^2 E^\vee \otimes E^\vee \otimes \Omega_L^1 \longrightarrow E^\vee \otimes E^\vee \otimes \Omega_L^1 \longrightarrow E^\vee \otimes \Omega_L^1 \longrightarrow (E^\vee \otimes \Omega_L^1)|_Z \longrightarrow -1
\]

\[
\cdots \longrightarrow \wedge^2 E^\vee \otimes S^2 E^\vee \longrightarrow E^\vee \otimes S^2 E^\vee \longrightarrow S^2 E^\vee \longrightarrow S^2 E^\vee|_Z \longrightarrow -2
\]
Examples (v):

In the same way we can describe $\Omega^2_X$ as a module over the Koszul complex

\[ \cdots \to \wedge^2 E^\vee \otimes \Omega^2_L \to E^\vee \otimes \Omega^2_L \to \Omega^2_L \to \Omega^2_{L|Z} \to 0 \]

\[ \cdots \to \wedge^2 E^\vee \otimes E^\vee \otimes \Omega^1_L \to E^\vee \otimes E^\vee \otimes \Omega^1_L \to E^\vee \otimes \Omega^1_L \to (E^\vee \otimes \Omega^1_L)|_Z \to -1 \]

\[ \cdots \to \wedge^2 E^\vee \otimes S^2 E^\vee \to E^\vee \otimes S^2 E^\vee \to S^2 E^\vee \to S^2 E^\vee |_Z \to -2 \]

2 forms of degree $-1$
Examples (v):

In the same way we can describe $\Omega^2_X$ as a module over the Koszul complex

$$\cdots \to \wedge^2 E^\vee \otimes \Omega^2_L \to E^\vee \otimes \Omega^2_L \to \Omega^2_L \to \Omega^2_L|_Z \to 0$$

$$\cdots \to \wedge^2 E^\vee \otimes E^\vee \otimes \Omega^1_L \to E^\vee \otimes E^\vee \otimes \Omega^1_L \to E^\vee \otimes \Omega^1_L \to (E^\vee \otimes \Omega^1_L)|_Z \to -1$$

$$\cdots \to \wedge^2 E^\vee \otimes S^2 E^\vee \to E^\vee \otimes S^2 E^\vee \to S^2 E^\vee \to S^2 E^\vee|_Z \to -2$$

**Note:** The de Rham differential $d_{DR} : \Omega^1_X \to \Omega^2_X$ is the sum $d_{DR} = \nabla + \kappa$, where $\kappa$ is the Koszul contraction

$$\kappa : \wedge^a E^\vee \otimes S^b E^\vee \to \wedge^{a-1} E^\vee \otimes S^{b+1} E^\vee.$$
Examples (vi):

**Important Remark:** [Behrend] If $E = \Omega^1_L$ and so $s$ is a 1-form, then a 2-form of degree $-1$ corresponds to a pair of elements

\[
\alpha \in (\Omega^1_L)^\vee \otimes \Omega^2_L \quad \text{and} \quad \phi \in (\Omega^1_L)^\vee \otimes \Omega^1_L
\]
such that $[\nabla, s^b](\phi) = s^b(\alpha)$.

Take $\phi = \text{id} \in (\Omega^1_L)^\vee \otimes \Omega^1_L$. Suppose the local $\nabla$ is chosen so that $\nabla(\text{id}) = 0$ (i.e. $\nabla$ is torsion free). Then $[\nabla, s^b](\text{id}) = ds$.

**Conclusion:** The pair $(\alpha, \text{id})$ gives a 2-form of degree $-1$ iff $ds = s^b(\alpha)$, or equivalently $ds|_Z = 0$, i.e. is an almost closed 1-form on $L$. 
Twisted cotangent bundles (i):

Let $M$ be a complex algebraic manifold and let $(X, \omega)$ be the cotangent bundle of $M$ equipped with the standard symplectic form. This symplectic structure is uniquely characterized by the following properties:

1. The natural projection $\pi : X \to M$ is a smooth Lagrangian fibration.
2. For any locally defined one form $\alpha$ on $M$ we have $t_\alpha^*\omega = \omega + \pi^*(d\alpha)$.

Twisted cotangent bundles are symplectic structures that are modeled on this geometry.
Twisted cotangent bundles (ii):

**Definition:** A twisted cotangent bundle over $M$ is specified by data $(\pi_Y : Y \to M, \omega_Y)$, where

- $\pi_Y : Y \to M$ is a torsor over $T^\vee M$;
- $\omega_Y$ is an algebraic symplectic form on $Y$, and:
  - The projection $\pi_Y : Y \to M$ is a Lagrangian fibration for $\omega_Y$.
  - For any locally defined one form $\alpha$ on $M$ we have
    $$t^*_\alpha \omega_Y = \omega_Y + \pi_Y^*(d\alpha).$$

**Note:** The $T^\vee M$-torsor structure is superfluous. It is uniquely determined from $\pi_Y$ and $\omega_Y$. Indeed, the infinitesimal action of a local one form $\alpha$ is given by the vector field $\Theta_{\omega_Y}^{-1}(\pi_Y^*\alpha)$. 
Twisted cotangent bundles (iii):

**Recall:** Let $C^\bullet = \left[ C^0 \xrightarrow{d} C^1 \right]$ be a complex of sheaves of $\mathbb{C}$-vector spaces on $M$ concentrated in degrees 0 and 1. Then a torsor over $C^\bullet$ is a pair $(A, t)$, where $A$ is a $C^0$-torsor and $t : A \rightarrow C^1$ is a trivialization of the associated $C^1$-torsor $d(A)$. Concretely $t$ is a map of sheaves satisfying $t(a + c) = t(a) + d(c)$ for all $a \in A, c \in C^0$. 

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Twisted cotangent bundles (iii):

Recall: Let $C^\bullet = \left[ C^0 \xrightarrow{d} C^1 \right]$ be a complex of sheaves of $\mathbb{C}$-vector spaces on $M$ concentrated in degrees 0 and 1. Then a torsor over $C^\bullet$ is a pair $(A, t)$, where $A$ is a $C^0$-torsor and $t : A \to C^1$ is a trivialization of the associated $C^1$-torsor $d(A)$.

**Lemma: [Beilinson-Berstein]** There is a canonical equivalence of groupoids

$$\left( \text{twisted cotangent bundles over } M \right) \leftrightarrow \left( \Omega_{\tilde{M}}^{\geq 1}[1]-\text{torsors} \right)$$
Twisted cotangent bundles (iv):

The equivalence of groupoids is described as follows:

→ Given a twisted cotangent bundle \((\pi_Y : Y \to M, \omega_Y)\) we define a \(\Omega^\geq 1_M[1]\)-torsor \((A, c)\), where \(A\) is the sheaf of sections of \(\pi_Y\), and \(c : A \to \Omega^\geq 1_Y^{2,\text{cl}}\) is given by \(c(a) = a^* \omega_Y\).

← Conversely, given a \(\Omega^\geq 1_M[1]\)-torsor \((A, c)\), define a twisted cotangent bundle \((\pi_Y : Y \to M, \omega_Y)\) by taking \(\pi_Y : Y \to M\) to be the total space of the \(\Omega^1_M\)-torsor \(A\), and \(\omega_Y\) to be the unique form such that for every local section \(\sigma\) of \(\pi_Y\), the associated isomorphism of \(T^\vee M\)-torsors \(f_\sigma : Y \to T^\vee M\) satisfies \(f_\sigma^* (\omega + \pi^*(c(\sigma))) = \omega_Y\).