A brief survey of 2D K-HAs

Andrei Neguț

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6/4/2020
Let $Q$ be a quiver (oriented graph) with vertex set $I$. For a field $\mathbb{F}$ and a collection of integers $v = (v_i)_{i \in I} \in \mathbb{N}_0^I$, define:

$M_v = \bigoplus_{e=ij \text{ edge}} \text{Hom}(\mathbb{F}^{v_i}, \mathbb{F}^{v_j})$
Hall algebras for quivers, following Ringel and Green, I

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- There is an action of $GL_{\mathbf{v}} = \prod_{i \in I} GL_{v_i}(\mathbb{F})$ on $M_{\mathbf{v}}$ by conjugation, which allows us to define the quotient stack:

$$\mathcal{M}_{\mathbf{v}} = M_{\mathbf{v}} / GL_{\mathbf{v}}$$

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whose points are quiver representations $V$ mod isomorphism.

• We may construct the extension diagram:

$$\{ (0 \rightarrow V' \rightarrow W \rightarrow V \rightarrow 0) \}$$

$$\begin{array}{ccc}
\mathcal{M}_{\mathbf{v}} \times \mathcal{M}_{\mathbf{v}'} & \xrightarrow{\pi_1} & \mathcal{M}_{\mathbf{v}+\mathbf{v}'} \\
\mathcal{M}_{\mathbf{v}} \times \mathcal{M}_{\mathbf{v}'} & \xrightarrow{\pi_2} & \mathcal{M}_{\mathbf{v}+\mathbf{v}'}
\end{array}$$

where $\pi_1$ remembers $(V, V')$ and $\pi_2$ remembers $W$. 

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Let us consider the $\mathbb{C}$–vector space of functions:

$$H_Q = \bigoplus_{v \in \mathbb{N}_0^I} \text{Fun}(\mathcal{M}_v(\mathbb{F}), \mathbb{C})$$

where $\mathbb{F}$ is a finite field of cardinality $q$. 

The extension diagram gives rise to an associative product:

$$H_Q \otimes H_Q \to H_Q, \quad f \ast g = \pi_2! \left( \pi_1^* (f \boxdot g) \right) \cdot q$$

We call $(H_Q, \ast)$ the Hall algebra of the quiver $Q$.

Theorem (Ringel, Green): If $Q$ has no loops, there is an embedding:

$$U^+ \oplus \mathbb{Q}(Q) \ni \iota \mapsto H_Q, \quad e_i \mapsto 1_{M_i}$$

of algebras, where $\varsigma_i \in \mathbb{N}_0^I$ has 1 on the $i$–th position and 0 elsewhere.

Note that $\iota$ is an isomorphism if $Q$ is a finite Dynkin diagram.
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Let us consider the \( \mathbb{C} \)-vector space of functions:

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where \( \mathbb{F} \) is a finite field of cardinality \( q \).

The extension diagram gives rise to an associative product:

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U_q^+(Q) \xhookrightarrow{\iota} H_Q, \quad e_i \mapsto 1_{\mathcal{M}_{\varsigma_i}}
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The extension diagram gives rise to a coassociative coproduct:

\[ H_Q \xrightarrow{\Delta} H_Q \otimes H_Q, \quad \Delta(x) = \pi_1!(\pi_2^*(x)) \cdot q^{\ldots} \]
Hall algebras for quivers, following Ringel and Green, III

- The extension diagram gives rise to a coassociative coproduct:

\[ \Delta \colon H_Q \to H_Q \otimes H_Q, \quad \Delta(x) = \pi_1!(\pi_2^*(x)) \cdot q \cdots \]

- However, the product and coproduct are not compatible. To remedy the situation, one enlarges the Hall algebra:

\[ \tilde{H}_Q = H_Q \otimes_{\mathbb{C}} \mathbb{C}[\psi_{i}^{\pm 1}]_{i \in I} / (x\psi_i - \psi_i x) q \cdots \]
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\[ \tilde{H}_Q = H_Q \otimes \mathbb{C}[\psi_i^{\pm 1}]_{i \in I} \left/ (x \psi_i - \psi_i x q \cdot \ldots) \right. \]

and twists the coproduct by setting \( \tilde{\Delta}(\psi_i) = \psi_i \otimes \psi_i \) and:

\[
\text{if } \Delta(x) = \sum_a y_a \otimes z_a \text{ then } \tilde{\Delta}(x) = \sum_a y_a \psi_{\deg(z_a)} \otimes z_a
\]

where \( \deg(x) = v \) for \( x \in \text{Fun}(\mathcal{M}_v(\mathbb{F}), \mathbb{C}) \), and \( \psi_v = \prod_i \psi_i^{v_i} \).
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With this in mind, \( \iota \) becomes an embedding of bialgebras.
From 1 dimension to 2 dimensions

- The category of quiver representations has homological dimension 1, and thus behaves much like the category of coherent sheaves on a smooth curve (see arXiv:1802.09760 for an overview)

\[ M_v \Rightarrow N_v = T^* M_v \]

Explicitly, since \( M_v = M_v / \text{GL}_v \), then:

\[ N_v = \mu^{-1}(0) / \text{GL}_v \]

where \( \mu : M_v \oplus M_v^* \to \text{gl}_v \) is given by:

\[ \mu (X_e, Y_e) \]
From 1 dimension to 2 dimensions

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- If you want to study something 2-dimensional, akin to coherent sheaves on a smooth surface, one needs to replace:

\[ \mathcal{M}_v \rightsquigarrow \mathcal{N}_v = T^* \mathcal{M}_v \]

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- Explicitly, since \( \mathcal{M}_v = M_v / GL_v \), then:

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where \( \mu : M_v \oplus M_v^* \rightarrow \mathfrak{gl}_v \) is given by:

\[ \mu(X_e, Y_e)_{e \text{ edges of } Q} = \sum_e (X_e Y_e - Y_e X_e) \]
From Hall algebras to CoHAs/K-HAs

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- Below, we follow the construction of Schiffmann-Vasserot (which is “2D” in spirit) and note that there is a related framework due to Kontsevich-Soibelman (which is “3D” in spirit).
From Hall algebras to CoHAs/K-HAs

- Hall algebras involve the set of closed points of stacks of quiver representations, a basic numerical invariant.
- But there exist other (more refined) numerical invariants, for example the cohomology and $K$–theory of these stacks.
- Below, we follow the construction of Schiffmann-Vasserot (which is “2D” in spirit) and note that there is a related framework due to Kontsevich-Soibelman (which is “3D” in spirit).
- As in the “classical” case, consider the extension stack:

$$\{(0 \to V' \to W \to V \to 0)\}$$

$$\pi_1 : \mathcal{N}_V \times \mathcal{N}_{V'} \to \mathcal{N}_{V+V'}$$
$$\pi_2$$

where $\pi_1$ remembers $(V, V')$ and $\pi_2$ remembers $W$. Above, $V, V', W$ are double quiver representations which are killed by $\mu$. 

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A brief survey of 2D K-HAs
The K-HA

- Since $\mathcal{N}_v$ is a quasi-smooth derived stack, one may construct:

$$K\text{-HA}_Q = \bigoplus_{v \in \mathbb{N}_0^I} K_T(\mathcal{N}_v)$$

- The extension diagram gives rise to an associative product:

$$K\text{-HA}_Q \otimes K\text{-HA}_Q \rightarrow K\text{-HA}_Q, \quad f \ast g = \pi_2 \ast (\pi_1 (f \boxplus g))$$

- $(K\text{-HA}_Q, \ast)$ is called the K-theoretic Hall algebra of $Q$.

- This algebra (and its cohomological counterpart) was studied by Schiffmann–Vasserot, Davison, Yang–Zhao and many others.

- If $Q$ has no loops, there is an algebra homomorphism:

$$\mathbb{U}_q(\hat{Q}) \rightarrow K\text{-HA}_Q, \quad e_i \mapsto [L \otimes k]$$

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- where $T = \mathbb{C}^* \times \prod_{\text{edge}} \mathbb{C}_e^*$ acts on $\mathcal{N}_v$ as follows:
  - $\mathbb{C}^*$ acts by $t \cdot (..., X_{e'}, Y_{e'}, ...) \mapsto (..., X_{e'}, tY_{e'}, ...)$
  - $\mathbb{C}_e^*$ acts by $t \cdot (..., X_{e'}, Y_{e'}, ...) \mapsto (..., X_{e'} t^{\delta_e}, Y_{e'} t^{-\delta_e}, ...)$

- The extension diagram gives rise to an associative product:

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$$U^+_q(\hat{Q}) \rightarrow K-\text{HA}_Q, \quad e_{i,k} \mapsto [L \otimes^k] \text{ on } N_{\varsigma_i} = \text{point}/\mathbb{C}^*$$
Recall the closed embedding:

\[ \mathcal{N}_v = \mu^{-1}(0)/GL_v \xrightarrow{\iota} T^*M_v/GL_v \]
Equivariant $K$–theory

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- Since $T^*M_v$ is an affine space, we have:
  \[ K_T(T^*M_v/\text{GL}_v) \cong K_{T \times \text{GL}_v}(T^*M_v) \cong K_{T \times \text{GL}_v}(\text{point}) \]
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- If we let $\{ q, q_e \}_{e \text{ edge}}$ denote the elementary characters of $T$, and $\{ z_{ia} \}_{i \in I, 1 \leq a \leq v_i}$ denote the elementary characters of a maximal torus of $GL_v$ (which we henceforth fix), then we have:
  \[ K_{T \times GL_v}(\text{point}) \cong \mathbb{Z}[q_{\pm 1}, q_{e \pm 1}, \ldots, z_{i_1 \pm 1}, \ldots, z_{iv_i \pm 1}, \ldots]^W \]
  where $W = \prod_i \mathbb{S}(v_i)$ permutes the symbols $\{ z_{i_1}, \ldots, z_{iv_i} \}$. 
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  \[ K_{T \times GL_v}(\text{point}) \cong \mathbb{Z}[q^{\pm 1}, q_e^{\pm 1}, ..., z_{i_1}^{\pm 1}, ..., z_{i_{v_i}}^{\pm 1}, ...]^W \]
  where $W = \prod_i \mathcal{S}(v_i)$ permutes the symbols $\{z_{i_1}, ..., z_{i_{v_i}}\}$.

- By combining these ideas, the localization theorem implies:
  \[ K_T(N_v)_{\text{loc}} \overset{\text{loc}}{\longrightarrow} \mathbb{Q}(q, q_e, ..., z_{i_1}, ..., z_{i_{v_i}}, ...)^W \]
  where “loc” means “tensor with $\mathbb{Q}(q, q_e, ..., z_{i_1}, ..., z_{i_{v_i}}, ...)^W$”.

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Schiffmann-Vasserot noticed that, to make $\nu_*$ into an algebra homomorphism, the RHS must be interpreted as a shuffle algebra. The latter have been studied (in various contexts) by Feigin-Odesskii, Enriquez, Tsymbaliuk and many others.
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Shuffle algebras also appear in the Kontsevich-Soibelman framework of 3D CoHA’s, in a similar fashion, although we will not discuss this.
Schiffmann-Vasserot noticed that, to make $\iota_*$ into an algebra homomorphism, the RHS must be interpreted as a **shuffle algebra**. The latter have been studied (in various contexts) by Feigin-Odesskii, Enriquez, Tsymbaliuk and many others.

Shuffle algebras also appear in the Kontsevich-Soibelman framework of 3D CoHA’s, in a similar fashion, although we will not discuss this.

Explicitly, consider the following associative **shuffle product**:

$$V_Q = \bigoplus_{v \in \mathbb{N}_0^I} \mathbb{Q}(q, q_e, \ldots, z_{i1}, \ldots, z_{iv}, \ldots)^W$$

$$F \ast G = \text{Sym} \left[ F(z_{i1}, \ldots, z_{iv}) G(z_{i,v_i+1}, \ldots, z_{i,v_i+v'_i}) \prod_{i,i' \in I} \prod_{a \leq v_i, a' > v'_i} \zeta \left( \frac{Z_{ia}}{Z_{i'a'}} \right) \right]$$
The shuffle algebra I

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- Explicitly, consider the following associative shuffle product:

\[
\mathcal{V}_Q = \bigoplus_{v \in \mathbb{N}_0^I} Q(q, q_e, \ldots, z_{i1}, \ldots, z_{iv_i}, \ldots)^W
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\]

- and \( \zeta \left( \frac{z_{ia}}{z_{i'a'}} \right) = \left[ \frac{z_{i'a'} - z_{ia}}{z_{i'a'} - z_{ia}} \right]^{\delta_{i',i}} \prod_{e=ii'} \left( 1 - \frac{z_{ia}}{z_{i'a'} q_e} \right) \prod_{e=i'j'} \left( 1 - \frac{z_{ia}q_e}{z_{i'a'} q} \right) \)
The shuffle algebra II

- With this in mind, $K\text{-HA}_{Q,\text{loc}} \xrightarrow{b^*} \mathcal{V}_Q$ is an algebra isomorphism.
The shuffle algebra II

- With this in mind, $K\text{-HA}_{Q,\text{loc}} \xrightarrow{\sim} \mathcal{V}_Q$ is an algebra isomorphism.
- Besides being very explicit, the shuffle algebra point of view allows us to put a (topological) coproduct on an extension of $K\text{-HA}_{Q,\text{loc}}$. 

Theorem (Negut, for the cyclic quiver, Yang-Zhao in general) Let:

$$\tilde{\mathcal{V}}_Q = \langle \psi_i, k \rangle_{i \in I}^{k \geq 0} / \left( F(z^{ia}) \psi_j(w) - \psi_j(w) F(z^{ia}) \prod_{i, a} \zeta(z^{ia}w) \zeta(wz^{ia}) \right)$$

where $\psi_j(w) = \sum_{k=0}^{\infty} \psi_{j,k} w^k$. The following makes $\tilde{\mathcal{V}}_Q$ a bialgebra:

$$\Delta(\psi_i(z)) = \psi_i(z) \otimes \psi_i(z), \quad \Delta(F(z^{i_1}, \ldots, z^{i_v})) = \sum_{0 \leq w_i \leq v_i} \prod_{a} \psi_{i_{a}}(z^{a}) \otimes 1 \otimes \prod_{i, a} \zeta(z^{ia}w_{i+1}) \zeta(z^{i_{a+1}}, w_{i+1}, \ldots, z^{i_v})$$

Upshot: we must enlarge the shuffle algebra to make it a bialgebra.

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The shuffle algebra II

- With this in mind, $K\text{-HA}_{Q,\text{loc}} \overset{b_\ast}{\cong} \mathcal{V}_Q$ is an algebra isomorphism.
- Besides being very explicit, the shuffle algebra point of view allows us to put a (topological) coproduct on an extension of $K\text{-HA}_{Q,\text{loc}}$.
- **Theorem** (Neguț for the cyclic quiver, Yang-Zhao in general) Let:

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\tilde{\mathcal{V}}_Q = \left\langle \mathcal{V}_Q, \psi_{i,k} \right\rangle_{k \geq 0} / \left( F(z_{ia})\psi_j(w) - \psi_j(w) F(z_{ia}) \prod_{i,a} \frac{\zeta \left( \frac{z_{ia}}{w} \right)}{\zeta \left( \frac{w}{z_{ia}} \right)} \right)
\]

where $\psi_j(w) = \sum_{k=0}^{\infty} \frac{\psi_{j,k}}{w^k}$. The following makes $\tilde{\mathcal{V}}_Q$ a bialgebra:

\[
\Delta(\psi_i(z)) = \psi_i(z) \otimes \psi_i(z), \quad \Delta(F(z_{i1}, \ldots, z_{iv})) =
\]

\[
= \sum_{\{0 \leq w_i \leq v_i\}_{i \in I}} \left[ \prod_{a=w_i+1}^{v_i} \psi_i(z_{ia}) \otimes 1 \right] \left( \frac{F(z_{i1}, \ldots, z_{iw_i} \otimes z_{i,w_i+1}, \ldots, z_{iv})}{\prod_{i,i' \in I} \prod_{a \leq w_i,a' > w_i} \zeta \left( \frac{z_{i'a'}}{z_{ia}} \right)} \right)
\]

- Upshot: we must enlarge the shuffle algebra to make it a bialgebra.
• With this in mind, $\text{K-HA}_{Q,\text{loc}} \overset{\sim}{\rightleftharpoons} V_Q$ is an algebra isomorphism.
• Besides being very explicit, the shuffle algebra point of view allows us to put a (topological) coproduct on an extension of $\text{K-HA}_{Q,\text{loc}}$.
• **Theorem** (Neguț for the cyclic quiver, Yang-Zhao in general) Let:

$$
\widetilde{V}_Q = \left\langle V_Q, \psi_{i,k} \right\rangle_{k \geq 0}^{i \in I} / \left( F(z_{ia})\psi_j(w) - \psi_j(w)F(z_{ia}) \prod_{i,a} \frac{\zeta\left(\frac{z_{ia}}{w}\right)}{\zeta\left(\frac{w}{z_{ia}}\right)} \right)
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where $\psi_j(w) = \sum_{k=0}^{\infty} \frac{\psi_{j,k}}{w^k}$. The following makes $\widetilde{V}_Q$ a bialgebra:

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\Delta(\psi_i(z)) = \psi_i(z) \otimes \psi_i(z), \quad \Delta(F(z_{i1}, \ldots, z_{iv})) =
$$

$$
\sum_{\{0 \leq w_i \leq v_i\} \in I} \left[ \prod_{a=w_i+1}^{v_i} \psi_i(z_{ia}) \otimes 1 \right] \frac{F(z_{i1}, \ldots, z_{iw_i} \otimes z_{i,w_i+1}, \ldots, z_{iv})}{\prod_{i,i' \in I} \prod_{a \leq w_i, a' > w_i} \zeta\left(\frac{z_{i'a'}}{z_{ia}}\right)}
$$

• Upshot: we must enlarge the shuffle algebra to make it a bialgebra.
Let $Q$ be the Jordan quiver (one vertex, one loop). In this case, $N_d$ is the moduli stack of length $d$ sheaves on the affine plane.

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Shuffle algebras for surfaces I

• Let $Q$ be the Jordan quiver (one vertex, one loop). In this case, $\mathcal{N}_d$ is the moduli stack of length $d$ sheaves on the affine plane.

• To generalize this setting, consider the moduli stack $\mathcal{N}^S_d$ of length $d$ sheaves on any smooth surface $S$. Can one make:

$$K\text{-HA}_S = \bigoplus_{d=0}^{\infty} K(\mathcal{N}^S_d)$$

into an algebra?

\[\text{Definition (Negut)}\]

The shuffle algebra associated to $S$ is:

$$V_S = \bigoplus_{d=0}^{\infty} K^S(d(S/C^*) \times \ldots \times S/C^*)_{\text{loc}} F \ast G = \text{Sym} \left[ \left( F \ast G \right)_{1 \leq a \leq d} \prod_{d' < a'} \zeta_{aa'}(x) \right]$$

for all $F \in K((S/C^*)^d)$ and $G \in K((S/C^*)^{d'})$, where $\zeta_{aa'}(x)$ denotes the generator of the $a$-th factor of $K((\text{point}/C^*)^d)$. 

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- **Definition** (Neguţ) The shuffle algebra associated to $S$ is:

$$\mathcal{V}_S = \bigoplus_{d=0}^{\infty} K_{S(d)} \left( \bigotimes_{\text{d factors}} (S/\mathbb{C}^* \times \ldots \times S/\mathbb{C}^*) \right)_{\text{loc}}$$

$$F \ast G = \text{Sym} \left[ (F \boxtimes G) \prod_{1 \leq a \leq d} \prod_{d < a' \leq d + d'} \zeta_{aa'} \left( \frac{z_a}{z_{a'}} \right) \right]$$

$\forall F \in K((S/\mathbb{C}^*)^d)$, $G \in K((S/\mathbb{C}^*)^{d'})$, where $\zeta_{aa'}(x) = \wedge^\bullet (-x \cdot \mathcal{O}_{\Delta_{aa'}})$ and $z_a$ denotes the generator of the $a$–th factor of $K(\text{point}/\mathbb{C}^*)$. 

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Theorem (Yu Zhao) There is an algebra structure on $K-HA_S$, defined as $\pi_2^* \pi_1^!$ with the notations in Schiffmann-Vasserot’s extension diagram, where the $!$ denotes an appropriately defined refined pull-back morphism. Moreover, there exists a map:

$$(S/\mathbb{C}^*)^d \xrightarrow{\iota} \mathcal{N}_d^S$$

and taking the inverse of the localization homomorphism:

$$K-HA_{S,\text{loc}} \xrightarrow{(\iota_*)^{-1}} \mathcal{N}_S$$

provides an isomorphism with the shuffle algebra.
Theorem (Yu Zhao) There is an algebra structure on $K$-$\text{HA}_S$, defined as $\pi_2^* \pi_1^!$ with the notations in Schiffmann-Vasserot’s extension diagram, where the $!$ denotes an appropriately defined refined pull-back morphism. Moreover, there exists a map:

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Kapranov-Vasserot provided an independent construction of $K$-$\text{HA}_S$, which also accounts for higher dimensional sheaves.
• **Theorem** (Yu Zhao) There is an algebra structure on $\text{K-HA}_S$, defined as $\pi_2^*\pi_1^!$ with the notations in Schiffmann-Vasserot’s extension diagram, where the $!$ denotes an appropriately defined refined pull-back morphism. Moreover, there exists a map:

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• Kapranov-Vasserot provided an independent construction of $\text{K-HA}_S$, which also accounts for higher dimensional sheaves.

• Porta-Sala extended the constructions above to $\infty$–categories of complexes of coherent sheaves on the derived stacks in question.